Handbook of Convergence Theorems for (Stochastic) Gradient Methods

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Abstract

This is a handbook of simple proofs of the convergence of gradient and stochastic gradient descent type methods. We consider functions that are Lipschitz, smooth, convex, strongly convex, and/or Polyak-Lojasiewicz functions. Our focus is on "good proofs" that are also simple. Each section can be consulted separately. We start with proofs of gradient descent, then on stochastic variants, including minibatching and momentum. Then move on to nonsmooth problems with the subgradient method, the proximal gradient descent and their stochastic variants. Our focus is on global convergence rates and complexity rates. Some slightly less common proofs found here include that of SGD (Stochastic gradient descent) with a proximal step in 11, with momentum in Section 7, and with mini-batching in Section 6.

1 Introduction

Here we collect our favourite convergence proofs for gradient and stochastic gradient based methods. Our focus has been on simple proofs, that are easy to copy and understand, and yet achieve the best convergence rate for the setting.

Disclaimer: Theses notes are not proper review of the literature. Our aim is to have an easy to reference handbook. Most of these proofs are not our work, but rather a collection of known proofs. If you find these notes useful, feel free to cite them, but we kindly ask that you cite the original sources as well that are given either before most theorems or in the bibliographic notes at the end of each section.

How to use these notes

We recommend searching for the theorem you want in the table of contents, or in the in Table 1a just below, then going directly to the section to see the proof. You can then follow the hyperlinks for the assumptions and properties backwards as needed. For example, if you want to know about the proof of Gradient Descent in the convex and smooth case you can jump ahead to Section 3.1. There you will find you need a property of convex function given in Lemma 2.8. These notes were not made to be read linearly: it would be impossibly boring.

Contents

1	Intr	roduction	1
2	The	eory : Smooth functions and convexity	5
	2.1	Differentiability	5
		2.1.1 Notations	5
		2.1.2 Lipschitz functions	5
	2.2	Convexity	6
	2.3	Strong convexity	7
	2.4	Polyak-Łojasiewicz	8
	2.5	Smoothness	11
		2.5.1 Smoothness and nonconvexity	11
		2.5.2 Smoothness and Convexity	12
3	Gra	dient Descent	13
	3.1	Convergence for convex and smooth functions	13
	3.2	Convergence for strongly convex and smooth functions	16
	3.3	Convergence for Polyak-Lojasiewicz and smooth functions	17
	3.4	Bibliographic notes	18
4	The	eory : Sum of functions	18
	4.1	Definitions	18
	4.2	Expected smoothness	19
	4.3	Controlling the variance	20
		4.3.1 Interpolation	20
		4.3.2 Interpolation constants	21
		4.3.3 Variance transfer	23
5	Sto	chastic Gradient Descent	24
	5.1	Convergence for convex and smooth functions	24
	5.2	Convergence for strongly convex and smooth functions	28
	5.3	Convergence for Polyak-Lojasiewicz and smooth functions	29

	5.4	Bibliographic notes	30
6	Mir	hibatch SGD	31
	6.1	Definitions	31
	6.2	Convergence for convex and smooth functions	33
	6.3	Rates for strongly convex and smooth functions	34
	6.4	Bibliographic Notes	36
7	Sto	chastic Momentum	36
	7.1	The many ways of writing momentum	36
	7.2	Convergence for convex and smooth functions	37
	7.3	Bibliographic notes	39
8	The	eory : Nonsmooth functions	39
	8.1	Real-extended valued functions	39
	8.2	Subdifferential of nonsmooth convex functions	40
	8.3	Nonsmooth strongly convex functions	42
	8.4	Proximal operator	42
	8.5	Controlling the variance	44
9	Sto	chastic Subgradient Descent	46
	9.1	Convergence for convex functions and bounded gradients	47
	9.2	Convergence for strongly convex functions and bounded gradients	49
	9.3	Better convergence rates for convex functions with bounded solution	50
	9.4	Bibliographic notes	52
10	Pro	ximal Gradient Descent	52
	10.1	Convergence for convex functions	53
	10.2	Convergence for strongly convex functions	55
	10.3	Bibliographic notes	56
11	Sto	chastic Proximal Gradient Descent	56
	11.1	Complexity for convex functions	57
	11.2	Complexity for strongly convex functions	61
	11.3	Bibliographic notes	63
A	App	pendix	66
	A.1	Lemmas for Complexity	66

Table 1: Where to find the corresponding theorem and complexity for all the algorithms and assumptions. GD = Gradient Descent, SGD = Stochastic Gradient Descent, mini-SGD = SGD with mini-batching, momentum = SGD with momentum, also known as stochastic heavy ball, prox-GD is proximal GD, and proximal SGD. The X's are settings which are currently not covered in the handbook.

	conv	/ex	μ -strongly convex	μ –PL
Methods	L-smooth	G–Lipschitz	L-smooth	L-smooth
GD (16)	Theorem 3.4	Theorem 9.5	Theorem 3.6	Theorem 3.9
SGD (39)	Theorem 5.5	Theorem 9.5	Theorem 5.7	Theorem 5.9
mini-SGD (52)	Theorem 6.8	Х	Theorem 6.11	Х
momentum (58)	Theorem 7.4	Х	Х	Х
prox-GD (87)	Theorem 10.3	Х	Theorem 10.5	Х
prox-SGD (100)	Corollary 11.7	Х	Theorem 11.9	Х

(a) Theorems and Corollaries for each method

	convex		μ -strongly convex	μ –PŁ
Methods	L-smooth	G–Lipschitz	L-smooth	L-smooth
GD (16)	$\frac{L}{\epsilon}$	$\frac{G}{\epsilon^2}$	$\frac{L}{\mu}\log\left(\frac{1}{\epsilon}\right)$	$\frac{L}{\mu}\log\left(\frac{1}{\epsilon}\right)$
SGD (39)	$\frac{1}{\epsilon^2} \left(L_{\max} D^2 + \frac{\sigma_f^*}{L_{\max}} \right)^2$	$\frac{G}{\epsilon^2}$	$\max\left\{\frac{1}{\epsilon}\frac{\sigma_f^*}{\mu^2}, \ \frac{L_{\max}}{\mu}\right\}$	$\frac{\Delta_f^*}{\epsilon} \frac{L^2}{\mu^2}$
mini-SGD (52)	$\frac{1}{\epsilon^2} \left(\mathcal{L}_b D^2 + \frac{\sigma_b^*}{\mathcal{L}_b} \right)^2$	Х	$\max\left\{\frac{1}{\epsilon}\frac{\sigma_b^*}{\mu^2}, \ \frac{\mathcal{L}_b}{\mu}\right\}$	Х
momentum (58)	$\frac{1}{\epsilon^2} \frac{1}{4L^2} \left(D^2 + \sigma_f^* \right)^2$	Х	X	Х
prox-GD (87)	$\frac{L}{\epsilon}$	Х	$\frac{\frac{L}{\mu}\log\left(\frac{1}{\epsilon}\right)}{\max\left\{\frac{1}{\epsilon}\frac{\sigma_{F}^{*}}{E} \underline{L}_{\max}\right\}}$	Х
prox-SGD (100)	$\frac{\sigma_F^2}{\epsilon^2} \left(D^2 + \frac{\delta_F}{L} \right)$	Х	$\max\left\{\frac{1}{\epsilon}\frac{\sigma_F^*}{\mu^2}, \frac{L_{\max}}{\mu}\right\}$	Х

(b) The complexity of each algorithm, where $\epsilon > 0$ is given, and in each cell we given the number of iterations required to make $\mathbb{E}\left[\|x^{t+1} - x^*\|^2\right] \leq \epsilon$ in the strongly convex, $\mathbb{E}\left[f(x^t) - \inf f\right] \leq \epsilon$ in the convex setting and PL. Further σ_f^* is defined in (36), Δ_f^* in (35), $D^2 := \|x^0 - x^*\|^2$, $\delta_f := f(x^0) - \inf f$. For composite functions F = f + g we have $\delta_F := F(x^0) - \inf F$ and σ_F^* is defined in (72). For the mini-batch results with mini-batch size $b \in \mathbb{N}$ we have σ_b^* defined in (55) and \mathcal{L}_b defined in (54).

2 Theory : Smooth functions and convexity

2.1 Differentiability

2.1.1 Notations

Definition 2.1 (Jacobian). Let $\mathcal{F} : \mathbb{R}^d \to \mathbb{R}^p$ be differentiable, and $x \in \mathbb{R}^d$. Then we note $D\mathcal{F}(x)$ the **Jacobian** of \mathcal{F} at x, which is the matrix defined by its first partial derivatives:

$$[D\mathcal{F}(x)]_{ij} = \frac{\partial f_i}{\partial x_j}(x), \text{ for } i = 1, \dots, p, \ j = 1, \dots, d,$$

where we write $\mathcal{F}(x) = (f_1(x), \dots, f_p(x))$. Consequently $D\mathcal{F}(x)$ is a matrix with $D\mathcal{F}(x) \in \mathbb{R}^{p \times d}$.

Remark 2.2 (Gradient). If $f : \mathbb{R}^d \to \mathbb{R}$ is differentiable, then $Df(x) \in \mathbb{R}^{1 \times d}$ is a row vector, whose transpose is called the **gradient** of f at $x : \nabla f(x) = Df(x)^{\top} \in \mathbb{R}^{d \times 1}$.

Definition 2.3 (Hessian). Let $f : \mathbb{R}^d \to \mathbb{R}$ be twice differentiable, and $x \in \mathbb{R}^d$. Then we note $\nabla^2 f(x)$ the **Hessian** of f at x, which is the matrix defined by its second-order partial derivatives:

$$\left[\nabla^2 f(x)\right]_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}(x), \quad \text{for } i, j = 1, \dots, d.$$

Consequently $\nabla^2 f(x)$ is a $d \times d$ matrix.

Remark 2.4 (Hessian and eigenvalues). If f is twice differentiable, then its Hessian is always a symmetric matrix (Schwarz's Theorem). Therefore, the Hessian matrix $\nabla^2 f(x)$ admits deigenvalues (Spectral Theorem).

2.1.2 Lipschitz functions

Definition 2.5. Let $\mathcal{F} : \mathbb{R}^d \to \mathbb{R}^p$, and L > 0. We say that \mathcal{F} is *L*-Lipschitz if

for all
$$x, y \in \mathbb{R}^d$$
, $\|\mathcal{F}(y) - \mathcal{F}(x)\| \le L \|y - x\|$.

A differentiable function is L-Lipschitz if and only if its differential is uniformly bounded by L.

Lemma 2.6. Let $\mathcal{F} : \mathbb{R}^d \to \mathbb{R}^p$ be differentiable, and L > 0. Then \mathcal{F} is *L*-Lipschitz if and only if

for all
$$x \in \mathbb{R}^d$$
, $||D\mathcal{F}(x)|| \le L$

Proof. \Rightarrow Assume that \mathcal{F} is *L*-Lipschitz. Let $x \in \mathbb{R}^d$, and let us show that $||D\mathcal{F}(x)|| \leq L$. This is equivalent to show that $||D\mathcal{F}(x)v|| \leq L$, for any $v \in \mathbb{R}^d$ such that ||v|| = 1. For a given v, the directional derivative is given by

$$D\mathcal{F}(x)v = \lim_{t\downarrow 0} \frac{\mathcal{F}(x+tv) - \mathcal{F}(x)}{t}.$$

Taking the norm in this equality, and using our assumption that \mathcal{F} is L-Lipschitz, we indeed obtain

$$\|D\mathcal{F}(x)v\| = \lim_{t \downarrow 0} \frac{\|\mathcal{F}(x+tv) - \mathcal{F}(x)\|}{t} \le \lim_{t \downarrow 0} \frac{L\|(x+tv) - x\|}{t} = \lim_{t \downarrow 0} \frac{Lt\|v\|}{t} = L.$$

 \Leftarrow Assume now that $\|D\mathcal{F}(z)\| \leq L$ for every vector $z \in \mathbb{R}^d$, and let us show that \mathcal{F} is *L*-Lipschitz. For this, fix $x, y \in \mathbb{R}^d$, and use the Mean-Value Inequality (see e.g. [8, Theorem 17.2.2]) to write

$$\|\mathcal{F}(y) - \mathcal{F}(x)\| \le \left(\sup_{z \in [x,y]} \|D\mathcal{F}(z)\|\right) \|y - x\| \le L \|y - x\|.$$

2.2 Convexity

Definition 2.7. We say that $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is convex if

for all
$$x, y \in \mathbb{R}^d$$
, for all $t \in [0, 1]$, $f(tx + (1 - t)y) \le tf(x) + (1 - t)f(y)$. (1)

The next two lemmas characterize the convexity of a function with the help of first and secondorder derivatives. These properties will be heavily used in the proofs.

Lemma 2.8. If $f : \mathbb{R}^d \to \mathbb{R}$ is convex and differentiable then,

for all
$$x, y \in \mathbb{R}^d$$
, $f(x) \ge f(y) + \langle \nabla f(y), x - y \rangle$. (2)

Proof. We can deduce (2) from (1) by dividing by t and re-arranging

$$\frac{f(y+t(x-y)) - f(y)}{t} \le f(x) - f(y).$$

Now taking the limit when $t \to 0$ gives

$$\langle \nabla f(y), x - y \rangle \le f(x) - f(y).$$

Lemma 2.9. Let $f : \mathbb{R}^d \to \mathbb{R}$ be convex and twice differentiable. Then, for all $x \in \mathbb{R}^d$, for every eigenvalue λ of $\nabla^2 f(x)$, we have $\lambda \ge 0$.

Proof. Since f is convex we can use (2) twice (permuting the roles of x and y) and summing the resulting two inequalities, to obtain that

for all
$$x, y \in \mathbb{R}^d$$
, $\langle \nabla f(y) - \nabla f(x), y - x \rangle \ge 0.$ (3)

Now, fix $x, v \in \mathbb{R}^d$, and write

$$\langle \nabla^2 f(x)v, v \rangle = \langle \lim_{t \to 0} \frac{\nabla f(x+tv) - \nabla f(x)}{t}, v \rangle = \lim_{t \to 0} \frac{1}{t^2} \langle \nabla f(x+tv) - \nabla f(x), (x+tv) - x \rangle \ge 0,$$

where the first equality follows because the gradient is a continuous function and the last inequality follows from (3). Now we can conclude : if λ is an eigenvalue of $\nabla^2 f(x)$, take any non zero eigenvector $v \in \mathbb{R}^d$ and write

$$\lambda \|v\|^2 = \langle \lambda v, v \rangle = \langle \nabla^2 f(x)v, v \rangle \ge 0.$$

Example 2.10 (Least-squares is convex). Let $\Phi \in \mathcal{M}_{n,d}(\mathbb{R})$ and $y \in \mathbb{R}^n$, and let $f(x) = \frac{1}{2} \|\Phi x - y\|^2$ be the corresponding least-squares function. Then f is convex, since $\nabla^2 f(x) \equiv \Phi^{\top} \Phi$ is positive semi-definite.

2.3 Strong convexity

Definition 2.11. Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, and $\mu > 0$. We say that f is μ -strongly convex if, for every $x, y \in \mathbb{R}^d$, and every $t \in [0, 1]$ we have that

$$\mu \frac{t(1-t)}{2} \|x-y\|^2 + f(tx + (1-t)y) \le tf(x) + (1-t)f(y).$$

We say that μ is the strong convexity constant of f.

The lemma below shows that it is easy to craft a strongly convex function : just add a multiple of $\|\cdot\|^2$ to a convex function. This happens for instance when using Tikhonov regularization (a.k.a. ridge regularization) in machine learning or inverse problems.

Lemma 2.12. Let $f : \mathbb{R}^d \to \mathbb{R}$, and $\mu > 0$. The function f is μ -strongly convex if and only if there exists a convex function $g : \mathbb{R}^d \to \mathbb{R}$ such that $f(x) = g(x) + \frac{\mu}{2} ||x||^2$.

Proof. Given f and μ , define $g(x) := f(x) - \frac{\mu}{2} ||x||^2$. We need to prove that f is μ -strongly convex if and only if g is convex. We start from Definition 2.11 and write (we note $z_t = (1-t)x + ty$):

f is μ -strongly convex

$$\Leftrightarrow \quad \forall t \; \forall x, y, \quad f(z_t) + \frac{\mu}{2} t(1-t) \|x - y\|^2 \le (1-t)f(x) + tf(y)$$

$$\Leftrightarrow \quad \forall t \; \forall x, y, \quad g(z_t) + \frac{\mu}{2} \|z_t\|^2 + \frac{\mu}{2} t(1-t) \|x - y\|^2 \le (1-t)g(x) + tg(y) + (1-t)\frac{\mu}{2} \|x\|^2 + t\frac{\mu}{2} \|y\|^2.$$
Let us now getter all the terms multiplied by μ to find that

Let us now gather all the terms multiplied by μ to find that

$$\begin{aligned} & \frac{1}{2} \|z_t\|^2 + \frac{1}{2} t(1-t) \|x-y\|^2 - (1-t) \frac{1}{2} \|x\|^2 - t \frac{1}{2} \|y\|^2 \\ &= (1-t)^2 \|x\|^2 + t^2 \|y\|^2 + 2t(1-t) \langle x, y \rangle + t(1-t) \|x\|^2 + t(1-t) \|y\|^2 - 2t(1-t) \langle x, y \rangle \\ &- (1-t) \|x\|^2 - t \|y\|^2 \\ &= \|x\|^2 \left((1-t)^2 + t(1-t) - (1-t) \right) + \|y\|^2 \left(t^2 + t(1-t) - t \right) \\ &= 0. \end{aligned}$$

So we see that all the terms in μ disappear, and what remains is exactly the definition for g to be convex. \Box

Lemma 2.13. If $f : \mathbb{R}^d \to \mathbb{R}$ is a continuous strongly convex function, then f admits a unique minimizer.

Proof. See [33, Corollary 2.20].

Now we present some useful variational inequalities satisfied by strongly convex functions.

Lemma 2.14. If $f : \mathbb{R}^d \to \mathbb{R}$ is μ -strongly convex and differentiable function then

for all
$$x, y \in \mathbb{R}^d$$
, $f(y) \ge f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} ||y - x||^2$. (4)

Proof. Define $g(x) := f(x) - \frac{\mu}{2} ||x||^2$. According to Lemma 2.12, g is convex. It is also clearly differentiable by definition. According to the sum rule, we have $\nabla f(x) = \nabla g(x) + \mu x$. Therefore we can use the convexity of g with Definition 8.4 to write

$$f(y) - f(x) - \langle \nabla f(x), y - x \rangle \ge \frac{\mu}{2} \|y\|^2 - \frac{\mu}{2} \|x\|^2 - \langle \mu x, y - x \rangle = \frac{\mu}{2} \|y - x\|^2.$$

Lemma 2.15. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a twice differentiable μ -strongly convex function. Then, for all $x \in \mathbb{R}^d$, for every eigenvalue λ of $\nabla^2 f(x)$, we have $\lambda \ge \mu$.

Proof. Define $g(x) := f(x) - \frac{\mu}{2} ||x||^2$, which is convex according to Lemma 2.12. It is also twice differentiable, by definition, and we have $\nabla^2 f(x) = \nabla^2 g(x) + \mu I d$. So the eigenvalues of $\nabla^2 f(x)$ are equal to the ones of $\nabla^2 g(x)$ plus μ . We can conclude by using Lemma 2.9.

Example 2.16 (Least-squares and strong convexity). Let f be a least-squares function as in Exercice 2.10. Then f is strongly convex if and only if Φ is injective. In this case, the strong convexity constant μ is $\lambda_{\min}(\Phi^{\top}\Phi)$, the smallest eigenvalue of $\Phi^{\top}\Phi$.

2.4 Polyak-Łojasiewicz

Definition 2.17 (Polyak-Lojasiewicz). Let $f : \mathbb{R}^d \to \mathbb{R}$ be differentiable, and $\mu > 0$. We say that f is μ -Polyak-Lojasiewicz if it is bounded from below, and if for all $x \in \mathbb{R}^d$

$$f(x) - \inf f \le \frac{1}{2\mu} \|\nabla f(x)\|^2.$$
 (5)

We just say that f is Polyak-Lojasiewicz (PL for short) if there exists $\mu > 0$ such that f is μ -Polyak-Lojasiewicz.

The Polyak-Lojasiewicz property is weaker than strong convexity, as we see next.

Lemma 2.18. Let $f : \mathbb{R}^d \to \mathbb{R}$ be differentiable, and $\mu > 0$. If f is μ -strongly convex, then f is μ -Polyak-Lojasiewicz.

Proof. Let x^* be a minimizer of f (see Lemma 2.13), such that $f(x^*) = \inf f$. Multiplying (4) by minus one and substituting $y = x^*$ as the minimizer, we have that

$$\begin{aligned} f(x) - f(x^*) &\leq \langle \nabla f(x), x - x^* \rangle - \frac{\mu}{2} \| x^* - x \|^2 \\ &= -\frac{1}{2} \| \sqrt{\mu} (x - x^*) - \frac{1}{\sqrt{\mu}} \nabla f(x) \|^2 + \frac{1}{2\mu} \| \nabla f(x) \|^2 \\ &\leq \frac{1}{2\mu} \| \nabla f(x) \|^2. \end{aligned}$$

It is important to note that the Polyak-Lojasiewicz property can hold without strong convexity or even convexity, as illustrated in the next examples.

Example 2.19 (Least-squares is PL). Let f be a least-squares function as in Example 2.10. Then it is a simple exercise to show that f is PL, and that the PL constant μ is $\lambda_{\min}^*(\Phi^{\top}\Phi)$, the smallest *nonzero* eigenvalue of $\Phi^{\top}\Phi$ (see e.g. [10, Example 3.7]).

Example 2.20 (Nonconvex PŁ functions).

- Let $f(t) = t^2 + 3\sin(t)^2$. It is an exercise to verify that f is PL, while not being convex (see Lemma A.3 for more details).
- If $\Omega \subset \mathbb{R}^d$ is a closed set and $f(x) = \operatorname{dist}(x; \Omega)^2$ is the squared distance function to this set, then it can be shown that f is PL. See Figure 1 for an example, and [9] for more details.

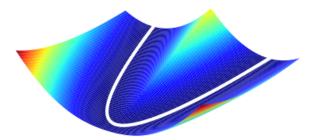


Figure 1: Graph of a PL function $f : \mathbb{R}^2 \to \mathbb{R}$. Note that the function is not convex, but that the only critical points are the global minimizers (displayed as a white curve).

Example 2.21 (PL for nonlinear models). Let $f(x) = \frac{1}{2} ||\Phi(x) - y||^2$, where $\Phi : \mathbb{R}^d \to \mathbb{R}^n$ is

differentiable. Then f is PL if $D\Phi^{\top}(x)$ is uniformly injective:

there exists
$$\mu > 0$$
 such that for all $x \in \mathbb{R}^d$, $\lambda_{\min}(D\Phi(x)D\Phi(x)^\top) \ge \mu$. (6)

Indeed it suffices to write

$$\|\nabla f(x)\|^{2} = \|D\Phi(x)^{\top}(\Phi(x) - y)\|^{2} \ge \mu \|\Phi(x) - y\|^{2} = 2\mu f(x) \ge 2\mu (f(x) - \inf f).$$

Note that assumption (6) requires $d \ge n$, which holds if Φ represents an overparametrized neural network. For more refined arguments, including less naive assumptions and exploiting the neural network structure of Φ , see [23].

One must keep in mind that the PL property is rather strong, as it is a *global* property and requires the following to be true, which is typical of convexity.

Lemma 2.22. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a differentiable PL function. Then $x^* \in \operatorname{argmin} f$ if and only if $\nabla f(x^*) = 0$.

Proof. Immediate from plugging in $x = x^*$ in (5).

Remark 2.23 (Local Łojasiewicz inequalities). In this document we focus only on the Polyak-Lojasiewicz inequality, for simplicity. Though there exists a much larger family of Lojasiewicz inequalities, which by and large cover most functions used in practice.

- The inequality can be more *local*. For instance by requiring that (5) holds only on some subset Ω ⊂ ℝ^d instead of the whole ℝ^d. For instance, logistic functions typically verify (5) on every bounded set, but not on the whole space. The same can be said about the empirical risk associated to wide enough neural networks [23].
- While PL describes functions that grow like $x \mapsto \frac{\mu}{2} ||x||^2$, there are *p-Lojasiewicz* inequalities describing functions that grow like $x \mapsto \frac{\mu^{p-1}}{p} ||x||^p$ and satisfy $f(x) \inf f \leq \frac{1}{q\mu} ||\nabla f(x)||^q$ on some set Ω , with $\frac{1}{p} + \frac{1}{q} = 1$.
- The inequality can be even more local, by dropping the property that every critical point is a global minimum. For this we do not look at the growth of $f(x) - \inf f$, but of $f(x) - f(x^*)$ instead, where $x^* \in \mathbb{R}^d$ is a critical point of interest. This can be written as

for all
$$x \in \Omega$$
, $f(x) - f(x^*) \le \frac{1}{q\mu} \|\nabla f(x)\|^q$, where $\frac{1}{p} + \frac{1}{q} = 1.$ (7)

A famous result [4, Corollary 16] shows that any semi-algebraic function (e.g. sums and products of polynomials by part functions) verifies (7) at every $x^* \in \mathbb{R}^d$ for some $p \ge 1$, $\mu > 0$, and Ω being an appropriate neighbourhood of x^* . This framework includes for instance quadratic losses evaluating a Neural Network with ReLU as activations.

2.5 Smoothness

Definition 2.24. Let $f : \mathbb{R}^d \to \mathbb{R}$, and L > 0. We say that f is *L*-smooth if it is differentiable and if $\nabla f : \mathbb{R}^d \to \mathbb{R}^d$ is *L*-Lipschitz:

for all
$$x, y \in \mathbb{R}^d$$
, $\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\|$. (8)

2.5.1 Smoothness and nonconvexity

As for the convexity (and strong convexity), we give two characterizations of the smoothness by means of first and second order derivatives.

Lemma 2.25. If $f : \mathbb{R}^d \to \mathbb{R}$ is *L*-smooth then

for all
$$x, y \in \mathbb{R}^d$$
, $f(y) \le f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2} ||y - x||^2$. (9)

Proof. Let $x, y \in \mathbb{R}^d$ be fixed. Let $\phi(t) := f(x + t(y - x))$. Using the Fundamental Theorem of Calculus on ϕ , we can write that

$$\begin{split} f(y) &= f(x) + \int_{t=0}^{1} \langle \nabla f(x+t(y-x)), y-x \rangle \, dt. \\ &= f(x) + \langle \nabla f(y), x-y \rangle + \int_{t=0}^{1} \langle \nabla f(x+t(y-x)) - \nabla f(x), y-x \rangle \, dt. \\ &\leq f(x) + \langle \nabla f(x), y-x \rangle + \int_{t=0}^{1} \| \nabla f(x+t(y-x)) - \nabla f(x) \| \| y-x \| dt \\ &\stackrel{(8)}{\leq} f(x) + \langle \nabla f(x), y-x \rangle + \int_{t=0}^{1} Lt \| y-x \|^2 \, dt \\ &\leq f(x) + \langle \nabla f(x), y-x \rangle + \frac{L}{2} \| y-x \|^2. \end{split}$$

Lemma 2.26. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a twice differentiable *L*-smooth function. Then, for all $x \in \mathbb{R}^d$, for every eigenvalue λ of $\nabla^2 f(x)$, we have $|\lambda| \leq L$.

Proof. Use Lemma 2.6 with $\mathcal{F} = \nabla f$, together with the fact that $D(\nabla f)(x) = \nabla^2 f(x)$. We obtain that, for all $x \in \mathbb{R}^d$, we have $\|\nabla^2 f(x)\| \leq L$. Therefore, for every eigenvalue λ of $\nabla^2 f(x)$, we can write for a nonzero eigenvector $v \in \mathbb{R}^d$ that

$$|\lambda| ||v|| = ||\lambda v|| = ||\nabla^2 f(x)v|| \le ||\nabla^2 f(x)|| ||v|| \le L ||v||.$$
(10)

The conclusion follows after dividing by $||v|| \neq 0$.

Remark 2.27. From Lemmas 2.25 and 2.14 we see that if a function is *L*-smooth and μ -strongly convex then $\mu \leq L$.

Some direct consequences of the smoothness are given in the following lemma. You can compare (12) with Lemma 2.18.

Lemma 2.28. If f is L-smooth and $\lambda > 0$ then

for all
$$x, y \in \mathbb{R}^d$$
, $f(x - \lambda \nabla f(x)) - f(x) \le -\lambda \left(1 - \frac{\lambda L}{2}\right) \|\nabla f(x)\|^2$. (11)

If moreover $\inf f > -\infty$, then

for all
$$x \in \mathbb{R}^d$$
, $\frac{1}{2L} \|\nabla f(x)\|^2 \le f(x) - \inf f.$ (12)

Proof. The first inequality (11) follows by inserting $y = x - \lambda \nabla f(x)$ in (9) since

$$\begin{aligned} f(x - \lambda \nabla f(x)) &\leq f(x) - \lambda \langle \nabla f(x), \nabla f(x) \rangle + \frac{L}{2} \|\lambda \nabla f(x)\|^2 \\ &= f(x) - \lambda \left(1 - \frac{\lambda L}{2}\right) \|\nabla f(x)\|^2. \end{aligned}$$

Assume now inf $f > -\infty$. By using (11) with $\lambda = 1/L$, we get (12) up to a multiplication by -1:

$$\inf f - f(x) \le f(x - \frac{1}{L}\nabla f(x)) - f(x) \le -\frac{1}{2L} \|\nabla f(x)\|^2. \quad \Box$$
(13)

2.5.2 Smoothness and Convexity

There are many problems in optimization where the function is both smooth and convex. Such functions enjoy properties which are strictly better than a simple combination of their convex and smooth properties.

Lemma 2.29. If $f : \mathbb{R}^d \to \mathbb{R}$ is convex and *L*-smooth, then for all $x, y \in \mathbb{R}^d$ we have that

$$\frac{1}{2L} \|\nabla f(y) - \nabla f(x)\|^2 \leq f(y) - f(x) - \langle \nabla f(x), y - x \rangle, \qquad (14)$$

$$\frac{1}{L} \|\nabla f(x) - \nabla f(y)\|^2 \leq \langle \nabla f(y) - \nabla f(x), y - x \rangle \quad \text{(Co-coercivity)}$$
(15)

Proof. To prove (14), fix $x, y \in \mathbb{R}^d$ and start by using the convexity and the smoothness of f to write, for every $z \in \mathbb{R}^d$,

$$f(x) - f(y) = f(x) - f(z) + f(z) - f(y)$$

$$\stackrel{(2)+(9)}{\leq} \langle \nabla f(x), x - z \rangle + \langle \nabla f(y), z - y \rangle + \frac{L}{2} ||z - y||^2.$$

To get the tightest upper bound on the right hand side, we can minimize the right hand side with respect to z, which gives

$$z = y - \frac{1}{L}(\nabla f(y) - \nabla f(x)).$$

Substituting this z in gives, after reorganizing the terms:

$$\begin{split} f(x) - f(y) &\leq \langle \nabla f(x), x - z \rangle + \langle \nabla f(y), z - y \rangle + \frac{L}{2} \|z - y\|^2. \\ &= \langle \nabla f(x), x - y \rangle - \frac{1}{L} \|\nabla f(y) - \nabla f(x)\|^2 + \frac{1}{2L} \|\nabla f(y) - \nabla f(x)\|^2 \\ &= \langle \nabla f(x), x - y \rangle - \frac{1}{2L} \|\nabla f(y) - \nabla f(x)\|^2. \end{split}$$

This proves (14). To obtain (15), apply (14) twice by interchanging the roles of x and y

$$\frac{1}{2L} \|\nabla f(y) - \nabla f(x)\|^2 \leq f(y) - f(x) - \langle \nabla f(x), y - x \rangle,$$

$$\frac{1}{2L} \|\nabla f(x) - \nabla f(y)\|^2 \leq f(x) - f(y) - \langle \nabla f(y), x - y \rangle,$$

and sum those two inequalities.

3 Gradient Descent

Problem 3.1 (Differentiable Function). We want to minimize a differentiable function $f : \mathbb{R}^d \to \mathbb{R}$. We require that the problem is well-posed, in the sense that argmin $f \neq \emptyset$.

Algorithm 3.2 (GD). Let $x^0 \in \mathbb{R}^d$, and let $\gamma > 0$ be a step size. The **Gradient Descent** (GD) algorithm defines a sequence $(x^t)_{t \in \mathbb{N}}$ satisfying

$$x^{t+1} = x^t - \gamma \nabla f(x^t). \tag{16}$$

Remark 3.3 (Vocabulary). Stepsizes are often called *learning rates* in the machine learning community.

We will now prove that the iterates of (GD) converge. In Theorem 3.4 we will prove sublinear convergence under the assumption that f is convex. In Theorem 3.6 we will prove linear convergence (a faster form of convergence) under the stronger assumption that f is μ -strongly convex.

3.1 Convergence for convex and smooth functions

Theorem 3.4. Consider the Problem (Differentiable Function) and assume that f is convex and L-smooth, for some L > 0. Let $(x^t)_{t \in \mathbb{N}}$ be the sequence of iterates generated by the (GD) algorithm, with a stepsize satisfying $0 < \gamma \leq \frac{1}{L}$. Then, for all $x^* \in \operatorname{argmin} f$, for all $t \in \mathbb{N}$ we

have that

$$f(x^t) - \inf f \le \frac{\|x^0 - x^*\|^2}{2\gamma t}.$$
 (17)

For this theorem we give two proofs. The first proof uses an energy function, that we will also use later on. The second proof is a direct proof taken from [5].

Proof of Theorem 3.4 with Lyapunov arguments. Let $x^* \in \operatorname{argmin} f$ be any minimizer of f. First, we will show that $f(x^t)$ is decreasing. Indeed we know from (11), and from our assumption $\gamma L \leq 1$, that

$$f(x^{t+1}) - f(x^t) \le -\gamma(1 - \frac{\gamma L}{2}) \|\nabla f(x^t)\|^2 \le 0.$$
(18)

Second, we will show that $||x^t - x^*||^2$ is also decreasing. For this we expand the squares to write

$$\frac{1}{2\gamma} \|x^{t+1} - x^*\|^2 - \frac{1}{2\gamma} \|x^t - x^*\|^2 = \frac{-1}{2\gamma} \|x^{t+1} - x^t\|^2 - \langle \nabla f(x^t), x^{t+1} - x^* \rangle \\
= \frac{-1}{2\gamma} \|x^{t+1} - x^t\|^2 - \langle \nabla f(x^t), x^{t+1} - x^t \rangle + \langle \nabla f(x^t), x^* - x^t \rangle.$$
(19)

Now to bound the right hand side we use the convexity of f and (2) to write

$$\langle \nabla f(x^t), x^* - x^t \rangle \le f(x^*) - f(x^t) = \inf f - f(x^t).$$

To bound the other inner product we use the smoothness of L and (9) which gives

$$-\langle \nabla f(x^t), x^{t+1} - x^t \rangle \le \frac{L}{2} \|x^{t+1} - x^t\|^2 + f(x^t) - f(x^{t+1}).$$

By using the two above inequalities in (19) we obtain

$$\frac{1}{2\gamma} \|x^{t+1} - x^*\|^2 - \frac{1}{2\gamma} \|x^t - x^*\|^2 \le \frac{-1}{2\gamma} \|x^{t+1} - x^t\|^2 - (f(x^{t+1}) - \inf f), \\
\le -(f(x^{t+1}) - \inf f).$$
(20)

Let us now combine the two positive decreasing quantities $f(x^t) - \inf f$ and $\frac{1}{2\gamma} ||x^t - x^*||^2$, and introduce the following Lyapunov energy, for all $t \in \mathbb{N}$:

$$E_t := \frac{1}{2\gamma} \|x^t - x^*\|^2 + t(f(x^t) - \inf f).$$

We want to show that it is decreasing with time. For this we start by writing

$$E_{t+1} - E_t = (t+1)(f(x^{t+1}) - f(x^t)) - t(f(x^t) - \inf f) + \frac{1}{2\gamma} \|x^{t+1} - x^*\|^2 - \frac{1}{2\gamma} \|x^t - x^*\|^2$$

$$= f(x^{t+1}) - \inf f + t(f(x^{t+1}) - f(x^t)) + \frac{1}{2\gamma} \|x^{t+1} - x^*\|^2 - \frac{1}{2\gamma} \|x^t - x^*\|^2.$$
(21)

Combining now (21), (18) and (20), we finally obtain (after cancelling terms) that

$$E_{t+1} - E_t \le f(x^{t+1}) - \inf f + \frac{1}{2\gamma} \|x^{t+1} - x^*\|^2 - \frac{1}{2\gamma} \|x^t - x^*\|^2 \qquad \text{using (18)}$$

$$\le f(x^{t+1}) - \inf f - (f(x^{t+1}) - \inf f) \qquad \text{using (20)}$$

$$\le 0.$$

Thus E_t is decreasing. Therefore we can write that

$$t(f(x^t) - \inf f) \le E_t \le E_0 = \frac{1}{2\gamma} ||x^0 - x^*||^2,$$

and the conclusion follows after dividing by t.

Proof of Theorem 3.4 with direct arguments. Let f be convex and L-smooth. It follows that

$$\begin{aligned} \|x^{t+1} - x^*\|^2 &= \|x^t - x^* - \frac{1}{L}\nabla f(x^t)\|^2 \\ &= \|x^t - x^*\|^2 - 2\frac{1}{L}\left\langle x^t - x^*, \nabla f(x^t)\right\rangle + \frac{1}{L^2}\|\nabla f(x^t)\|^2 \\ &\stackrel{(15)}{\leq} \|x^t - x^*\|^2 - \frac{1}{L^2}\|\nabla f(x^t)\|^2. \end{aligned}$$
(22)

Thus $||x^t - x^*||^2$ is a decreasing sequence in t, and thus consequently

$$\|x^{t} - x^{*}\| \le \|x^{0} - x^{*}\|.$$
(23)

Calling upon (11) and subtracting $f(x^*)$ from both sides gives

$$f(x^{t+1}) - f(x^*) \leq f(x^t) - f(x^*) - \frac{1}{2L} \|\nabla f(x^t)\|^2.$$
(24)

Applying convexity we have that

$$f(x^{t}) - f(x^{*}) \leq \langle \nabla f(x^{t}), x^{t} - x^{*} \rangle$$

$$\leq \|\nabla f(x^{t})\| \|x^{t} - x^{*}\| \stackrel{(23)}{\leq} \|\nabla f(x^{t})\| \|x^{0} - x^{*}\|.$$
(25)

Isolating $\|\nabla f(x^t)\|$ in the above and inserting in (24) gives

$$f(x^{t+1}) - f(x^*) \stackrel{(24)+(25)}{\leq} f(x^t) - f(x^*) - \underbrace{\frac{1}{2L} \frac{1}{\|x^0 - x^*\|^2}}_{\beta} (f(x^t) - f(x^*))^2 \tag{26}$$

Let $\delta_t = f(x^t) - f(x^*)$. Since $\delta_{t+1} \leq \delta_t$, and by manipulating (26) we have that

$$\delta_{t+1} \leq \delta_t - \beta \delta_t^2 \stackrel{\times \frac{1}{\delta_t \delta_{t+1}}}{\Leftrightarrow} \beta \frac{\delta_t}{\delta_{t+1}} \leq \frac{1}{\delta_{t+1}} - \frac{1}{\delta_t} \stackrel{\delta_{t+1} \leq \delta_t}{\Leftrightarrow} \beta \leq \frac{1}{\delta_{t+1}} - \frac{1}{\delta_t}.$$

Summing up both sides over t = 0, ..., T - 1 and using telescopic cancellation we have that

$$T\beta \leq \frac{1}{\delta_T} - \frac{1}{\delta_0} \leq \frac{1}{\delta_T}.$$

Re-arranging the above we have that

$$f(x^T) - f(x^*) = \delta_T \le \frac{1}{\beta(T-1)} = \frac{2L \|x^0 - x^*\|^2}{T}.$$

Corollary 3.5 ($\mathcal{O}(1/t)$ Complexity). Under the assumptions of Theorem 3.4, for a given $\epsilon > 0$ and $\gamma = L$ we have that

$$t \ge \frac{L}{\epsilon} \frac{\|x^0 - x^*\|^2}{2} \implies f(x^t) - \inf f \le \epsilon$$
(27)

3.2 Convergence for strongly convex and smooth functions

Now we prove the convergence of gradient descent for strongly convex and smooth functions.

Theorem 3.6. Consider the Problem (Differentiable Function) and assume that f is μ -strongly convex and L-smooth, for some $L \ge \mu > 0$. Let $(x^t)_{t \in \mathbb{N}}$ be the sequence of iterates generated by the (GD) algorithm, with a stepsize satisfying $0 < \gamma \le \frac{1}{L}$. Then, for $x^* = \operatorname{argmin} f$ and for all $t \in \mathbb{N}$:

$$\|x^{t+1} - x^*\|^2 \le (1 - \gamma\mu)^{t+1} \|x^0 - x^*\|^2.$$
(28)

Remark 3.7. Note that with the choice $\gamma = \frac{1}{L}$, the iterates enjoy a linear convergence with a rate of $(1 - \mu/L)$.

Below we provide two different proofs for this Theorem 3.6. The first one makes use of firstorder variational inequalities induced by the strong convexity and smoothness of f. The second one (assuming further that f is twice differentiable) exploits the fact that the eigenvalues of the Hessian of f are in between μ and L.

Proof of Theorem 3.6 with first-order properties. From (GD) we have that

$$\begin{aligned} \|x^{t+1} - x^*\|^2 &= \|x^t - x^* - \gamma \nabla f(x^t)\|^2 \\ &= \|x^t - x^*\|^2 - 2\gamma \left\langle \nabla f(x^t), x^t - x^* \right\rangle + \gamma^2 \|\nabla f(x^t)\|^2 \\ &\stackrel{(4)}{\leq} (1 - \gamma \mu) \|x^t - x^*\|^2 - 2\gamma (f(x^t) - \inf f) + \gamma^2 \|\nabla f(x^t)\|^2 \\ &\stackrel{(12)}{\leq} (1 - \gamma \mu) \|x^t - x^*\|^2 - 2\gamma (f(x^t) - \inf f) + 2\gamma^2 L(f(x^t) - \inf f) \\ &= (1 - \gamma \mu) \|x^t - x^*\|^2 - 2\gamma (1 - \gamma L) (f(x^t) - \inf f). \end{aligned}$$
(29)

Since $\frac{1}{L} \ge \gamma$ we have that $-2\gamma(1-\gamma L)$ is negative, and thus can be safely dropped to give

$$||x^{t+1} - x^*||^2 \le (1 - \gamma \mu) ||x^t - x^*||^2.$$

It now remains to unroll the recurrence.

Proof of Theorem 3.6 with the Hessian. Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be defined by $T(x) = x - \gamma \nabla f(x)$, so that we can write an iteration of Gradient Descent as $x^{t+1} = T(x^t)$. Note that the minimizer x^* verifies $\nabla f(x^*) = 0$, so it is a fixed point of T in the sense that $T(x^*) = x^*$. This means that $\|x^{t+1} - x^*\| = \|T(x^t) - T(x^*)\|$. Now we want to prove that

$$||T(x^{t}) - T(x^{*})|| \le (1 - \lambda \mu) ||x^{t} - x^{*}||.$$
(30)

Indeed, unrolling the recurrence from (30) would provide the desired bound (28).

We see that (30) is true as long as T is θ -Lipschitz, with $\theta = (1 - \lambda \mu)$. From Lemma 2.6, we know that is equivalent to proving that the norm of the differential of T is bounded by θ . It is easy to compute this differential : $DT(x) = I_d - \lambda \nabla^2 f(x)$. If we note $v_1(x) \leq \cdots \leq v_d(x)$ the eigenvalues of $\nabla^2 f(x)$, we know by Lemmas 2.15 and 2.26 that $\mu \leq v_i(x) \leq L$. Since we assume $\lambda L \leq 1$, we see that $0 \leq 1 - \lambda v_i(x) \leq 1 - \lambda \mu$. So we can write

for all
$$x \in \mathbb{R}^d$$
, $||DT(x)|| = \max_{i=1,\dots,d} |1 - \lambda v_i(x)| \le 1 - \lambda \mu = \theta$,

which allows us to conclude that (30) is true. To conclude the proof of Theorem 3.6, take the squares in (30) and use the fact that $\theta \in]0, 1[\Rightarrow \theta^2 \leq \theta$.

The linear convergence rate in Theorem 3.6 can be transformed into a complexity result as we show next.

Corollary 3.8 ($\mathcal{O}(\log(1/\epsilon))$) Complexity). Under the same assumptions as Theorem 3.6, for a given $\epsilon > 0$, we have that if $\gamma = 1/L$ then

$$t \ge \frac{L}{\mu} \log\left(\frac{1}{\epsilon}\right) \quad \Rightarrow \quad \|x^{t+1} - x^*\|^2 \le \epsilon \|x^0 - x^*\|^2.$$
(31)

The proof of this lemma follows by applying Lemma A.1 in the appendix.

3.3 Convergence for Polyak-Lojasiewicz and smooth functions

Here we present a convergence result for nonconvex functions satisfying the Polyak-Lojasiewicz condition (see Definition 2.17). This is a favorable setting, since all local minima and critical points are also global minima (Lemma 2.22), which will guarantee convergence. Moreover the PL property imposes a quadratic growth on the function, so we will recover bounds which are similar to the strongly convex case.

Theorem 3.9. Consider the Problem (Differentiable Function) and assume that f is μ -Polyak-Lojasiewicz and L-smooth, for some $L \ge \mu > 0$. Consider $(x^t)_{t \in \mathbb{N}}$ a sequence generated by the (GD) algorithm, with a stepsize satisfying $0 < \gamma \le \frac{1}{L}$. Then:

$$f(x^t) - \inf f \le (1 - \gamma \mu)^t (f(x^0) - \inf f).$$

Proof. We can use Lemma 2.25, together with the update rule of (GD), to write

$$\begin{aligned} f(x^{t+1}) &\leq f(x^{t}) + \langle \nabla f(x^{t}), x^{t+1} - x^{t} \rangle + \frac{L}{2} \|x^{t+1} - x^{t}\|^{2} \\ &= f(x^{t}) - \gamma \|\nabla f(x^{t})\|^{2} + \frac{L\gamma^{2}}{2} \|\nabla f(x^{t})\|^{2} \\ &= f(x^{t}) - \frac{\gamma}{2} (2 - L\gamma) \|\nabla f(x^{t})\|^{2} \\ &\leq f(x^{t}) - \frac{\gamma}{2} \|\nabla f(x^{t})\|^{2}, \end{aligned}$$

where in the last inequality we used our hypothesis on the stepsize that $\gamma L \leq 1$. We can now use the Polyak-Lojasiewicz property (recall Definition 2.17) to write:

$$f(x^{t+1}) \le f(x^t) - \gamma \mu(f(x^t) - \inf f).$$

The conclusion follows after subtracting $\inf f$ on both sides of this inequality, and using recursion.

Corollary 3.10 (log(1/ ϵ) Complexity). Under the same assumptions as Theorem 3.9, for a given $\epsilon > 0$, we have that if $\gamma = 1/L$ then

$$t \ge \frac{L}{\mu} \log\left(\frac{1}{\epsilon}\right) \quad \Rightarrow \quad f(x^t) - \inf f \le \epsilon \left(f(x^0) - \inf f\right).$$
 (32)

The proof of this lemma follows by applying Lemma A.1 in the appendix.

3.4 Bibliographic notes

Our second proof for convex and smooth in Theorem 3.4 is taken from [5]. Proofs in the convex and strongly convex case can be found in [31]. Our proof under the Polyak-Lojasiewicz was taken from [19].

4 Theory : Sum of functions

4.1 Definitions

In the next sections we will assume that our objective function is a sum of functions.

Problem 4.1 (Sum of Functions). We want to minimize a function $f : \mathbb{R}^d \to \mathbb{R}$ which writes as

$$f(x) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i=1}^{n} f_i(x),$$

where $f_i : \mathbb{R}^d \to \mathbb{R}$. We require that the problem is well-posed, in the sense that argmin $f \neq \emptyset$ and that the f_i 's are bounded from below.

Depending on the applications, we will consider two different sets of assumptions.

Assumption 4.2 (Sum of Convex). We consider the Problem (Sum of Functions) where each $f_i : \mathbb{R}^d \to \mathbb{R}$ is assumed to be convex.

Assumption 4.3 (Sum of L_{\max} -Smooth). We consider the Problem (Sum of Functions) where each $f_i : \mathbb{R}^d \to \mathbb{R}$ is assumed to be L_i -smooth. We will note $L_{\max} \stackrel{\text{def}}{=} \max_{1,\dots,n} L_i$, and $L_{\max} \stackrel{\text{def}}{=}$ $\frac{1}{n}\sum_{i=1}^{n}L_i$. We will also note L_f the Lipschitz constant of ∇f .

Note that, in the above Assumption (Sum of L_{max} -Smooth), the existence of L_f is not an assumption but the consequence of the smoothness of the f_i 's. Indeed:

Lemma 4.4. Consider the Problem (Sum of Functions). If the f_i 's are L_i -smooth, then f is L_{avg} -smooth.

Proof. Using the triangular inequality we have that

$$\begin{aligned} \|\nabla f(y) - \nabla f(x)\| &= \|\frac{1}{n} \sum_{i=1}^{n} \nabla f_i(y) - \nabla f_i(x)\| \leq \frac{1}{n} \sum_{i=1}^{n} \|\nabla f_i(y) - \nabla f_i(x)\| \leq \frac{1}{n} \sum_{i=1}^{n} L_i \|y - x\| \\ &= L_{\text{avg}} \|y - x\|. \end{aligned}$$

This proves that f(x) is L_{avg} -smooth.

Definition 4.5 (Notation). Given two random variables X, Y in \mathbb{R}^d , we note :

- the **expectation** of X as $\mathbb{E}[X]$,
- the expectation of X conditioned to Y as $\mathbb{E}[X \mid Y]$,
- the variance of X as $\mathbb{V}[X] \stackrel{\text{def}}{=} \mathbb{E}[||X \mathbb{E}[X]||^2].$

Lemma 4.6 (Variance and expectation). Let X be a random variable in \mathbb{R}^d .

- 1. For all $y \in \mathbb{R}^d$, $\mathbb{V}[X] \le \mathbb{E}\left[\|X y\|^2 \right]$.
- 2. $\mathbb{V}[X] \leq \mathbb{E}\left[\|X\|^2 \right]$.

Proof. Item 2 is a direct consequence of the first with y = 0. To prove item 1, we use that

$$||X - \mathbb{E}[X]||^{2} = ||X - y||^{2} + ||y - \mathbb{E}[X]||^{2} + 2\langle X - y, y - \mathbb{E}[X]\rangle,$$

and then take expectation to conclude

$$\mathbb{V}[X] = \mathbb{E}\left[\|X - y\|^2\right] - 2\mathbb{E}\left[\|y - \mathbb{E}[X]\|^2\right] \le \mathbb{E}\left[\|X - y\|^2\right].$$

4.2 Expected smoothness

Here we focus on the smoothness properties that the functions f_i verify in *expectation*. The socalled *expected smoothness* property below can be seen as "cocoercivity in expectation" (remember Lemma 2.29).

Lemma 4.7. If Assumptions (Sum of L_{max} -Smooth) and (Sum of Convex) hold, then f is L_{max} -smooth in expectation, in the sense that

for all
$$x, y \in \mathbb{R}^d$$
, $\frac{1}{2L_{\max}} \mathbb{E}\left[\|\nabla f_i(y) - \nabla f_i(x)\|^2 \right] \leq f(y) - f(x) - \langle \nabla f(x), y - x \rangle.$ (33)

Proof. Using (14) in Lemma 2.29 applied to f_i , together with the fact that $L_i \leq L_{\text{max}}$, allows us to write

$$\frac{1}{2L_{\max}} \|\nabla f_i(y) - \nabla f_i(x)\|^2 \leq f_i(y) - f_i(x) + \langle \nabla f_i(x), y - x \rangle.$$

To conclude, multiply the above inequality by $\frac{1}{n}$, and sum over *i*, using the fact that $\frac{1}{n}\sum_{i}f_{i}=f$ and $\frac{1}{n}\sum_{i}\nabla f_{i}=\nabla f$.

As a direct consequence we also have the analog of Lemma 2.29 in expectation.

Lemma 4.8. If Assumptions (Sum of L_{\max} -Smooth) and (Sum of Convex) hold, then, for every $x \in \mathbb{R}^d$ and every $x^* \in \operatorname{argmin} f$, we have that

$$\frac{1}{2L_{\max}} \mathbb{E}\left[\|\nabla f_i(x) - \nabla f_i(x^*)\|^2 \right] \leq f(x) - \inf f.$$
(34)

Proof. Apply Lemma 4.7 with $x = x^*$ and y = x, since $f(x^*) = \inf f$ and $\nabla f(x^*) = 0$.

4.3 Controlling the variance

Some stochastic problems are easier than others. For instance, a problem where all the f_i 's are the same is easy to solve, as it suffices to minimize one f_i to obtain a minimizer of f. We can also imagine that if the f_i are not exactly the same but look similar, the problem will also be easy. And of course, we expect that the easier the problem, the faster our algorithms will be. In this section we present one way to quantify this phenomena.

4.3.1 Interpolation

Definition 4.9. Consider the Problem (Sum of Functions). We say that interpolation holds if there exists a common $x^* \in \mathbb{R}^d$ such that $f_i(x^*) = \inf f_i$ for all i = 1, ..., n. In this case, we say that interpolation holds at x^* .

Even though unspecified, the x^* appearing in Definition 4.9 must be a minimizer of f.

Lemma 4.10. Consider the Problem (Sum of Functions). If interpolation holds at $x^* \in \mathbb{R}^d$, then $x^* \in \operatorname{argmin} f$.

Proof. Let interpolation hold at $x^* \in \mathbb{R}^d$. By Definition 4.9, this means that $x^* \in \operatorname{argmin} f_i$. Therefore, for every $x \in \mathbb{R}^d$,

$$f(x^*) = \frac{1}{n} \sum_{i=1}^n f_i(x^*) = \frac{1}{n} \sum_{i=1}^n \inf f_i \le \frac{1}{n} \sum_{i=1}^n f_i(x) = f(x).$$

This proves that $x^* \in \operatorname{argmin} f$.

Interpolation means that there exists some x^* that simultaneously achieves the minimum of all loss functions f_i . In terms of learning problems, this means that the model perfectly fits every data point. This is illustrated below with a couple of examples.

Example 4.11 (Least-squares and interpolation). Consider a regression problem with data $(\phi_i, y_i)_{i=1}^n \subset \mathbb{R}^d \times \mathbb{R}$, and let $f(x) = \frac{1}{2m} ||\Phi x - y||^2$ be the corresponding least-squares function with $\Phi = (\phi_i)_{i=1}^n$ and $y = (y_i)_{i=1}^n$. This is a particular case of Problem (Sum of Functions), with $f_i(x) = \frac{1}{2}(\langle \phi_i, x \rangle - y_i)^2$. We see here that interpolation holds if and only if there exists $x^* \in \mathbb{R}^d$ such that $\langle \phi_i, x^* \rangle = y_i$. In other words, we can find an hyperplane in $\mathbb{R}^d \times \mathbb{R}$ passing through each data point (ϕ_i, y_i) . This is why we talk about *interpolation*.

For this linear model, note that interpolation holds if and only if y is in the range of Φ , which is always true if Φ is surjective. This generically holds when d > n, which is usually called the *overparametrized* regime.

Example 4.12 (Neural Networks and interpolation). Let $\Phi : \mathbb{R}^d \to \mathbb{R}^n$, $y \in \mathbb{R}^n$, and consider the nonlinear least-squares $f(x) = \frac{1}{2} ||\Phi(x) - y||^2$. As in the linear case, interpolation holds if and only if there exists $x^* \in \mathbb{R}^d$ such that $\Phi(x^*) = y$, or equivalently, if $\inf f = 0$. The interpolation condition has drawn much attention recently because it was empirically observed that many overparametrized deep neural networks (with $d \gg n$) achieve $\inf f = 0$ [24, 23].

4.3.2 Interpolation constants

Here we introduce different measures of how far from interpolation we are. We start with a first quantity measuring how the infimum of f and the f_i 's are related.

Definition 4.13. Consider the Problem (Sum of Functions). We define the function noise as

$$\Delta_f^* \stackrel{\text{def}}{=} \inf f - \frac{1}{n} \sum_{i=1}^n \inf f_i.$$
(35)

Example 4.14 (Function noise for least-squares). Let f be a least-squares as in Example 4.11. It is easy to see that $\inf f_i = 0$, implying that the function noise is exactly $\Delta_f^* = \inf f$. We see in this case that $\Delta_f^* = 0$ if and only if interpolation holds (see also the next Lemma). If the function noise $\Delta_f^* = \inf f$ is nonzero, it can be seen as a measure of how far we are from interpolation.

Lemma 4.15. Consider the Problem (Sum of Functions). We have that

1. $\Delta_f^* \ge 0.$

2. Interpolation holds if and only if $\Delta_f^* = 0$.

Proof.

1. Let $x^* \in \operatorname{argmin} f$, so that we can write

$$\Delta_f^* = f(x^*) - \frac{1}{n} \sum_{i=1}^n \inf f_i \ge f(x^*) - \frac{1}{n} \sum_{i=1}^n f_i(x^*) = f(x^*) - f(x^*) = 0.$$

2. Let interpolation hold at $x^* \in \mathbb{R}^d$. According to Definition 4.9 we have $x^* \in \operatorname{argmin} f_i$. According to Lemma 4.10, we have $x^* \in \operatorname{argmin} f$. So we indeed have

$$\Delta_f^* = \inf f - \frac{1}{n} \sum_{i=1}^n \inf f_i = f(x^*) - \frac{1}{n} \sum_{i=1}^n f_i(x^*) = f(x^*) - f(x^*) = 0.$$

If instead we have $\Delta_f^* = 0$, then we can write for some $x^* \in \operatorname{argmin} f$ that

$$0 = \Delta_f^* = f(x^*) - \frac{1}{n} \sum_{i=1}^n \inf f_i = \frac{1}{n} \sum_{i=1}^n \left(f_i(x^*) - \inf f_i \right).$$

Clearly we have $f_i(x^*) - \inf f_i \ge 0$, so this sum being 0 implies that $f_i(x^*) - \inf f_i = 0$ for all i = 1, ..., n. In other words, interpolation holds.

We can also measure how close we are to interpolation using gradients instead of function values.

Definition 4.16. Let Assumption (Sum of L_{max} -Smooth) hold. We define the gradient noise as

$$\sigma_f^* \stackrel{\text{def}}{=} \inf_{x^* \in \operatorname{argmin} f} \mathbb{V} \left[\nabla f_i(x^*) \right], \tag{36}$$

where for a random vector $X \in \mathbb{R}^d$ we use

$$\mathbb{V}[X] := \mathbb{E}[\|X - \mathbb{E}[X]\|^2].$$

Lemma 4.17. Let Assumption (Sum of L_{max} -Smooth) hold. It follows that

- 1. $\sigma_f^* \ge 0$.
- 2. If Assumption (Sum of Convex) holds, then $\sigma_f^* = \mathbb{V}[\nabla f_i(x^*)]$ for every $x^* \in \operatorname{argmin} f$.
- 3. If interpolation holds then $\sigma_f^* = 0$. This becomes an equivalence if Assumption (Sum of Convex) holds.

Proof.

- 1. From Definition 4.5 we have that the variance $\mathbb{V}[\nabla f_i(x^*)]$ is nonnegative, which implies $\sigma_f^* \geq 0$.
- 2. Let $x^*, x' \in \operatorname{argmin} f$, and let us show that $\mathbb{V}[\nabla f_i(x^*)] = \mathbb{V}[\nabla f_i(x')]$. Since Assumptions (Sum of L_{\max} -Smooth) and (Sum of Convex) hold, we can use the expected smoothness via Lemma 4.8 to obtain

$$\frac{1}{2L_{\max}} \mathbb{E}\left[\|\nabla f_i(x') - \nabla f_i(x^*)\|^2 \right] \le f(x') - \inf f = \inf f - \inf f = 0.$$

This means that $\mathbb{E}\left[\|\nabla f_i(x') - \nabla f_i(x^*)\|^2\right] = 0$, which in turns implies that, for every $i = 1, \ldots, n$, we have $\|\nabla f_i(x') - \nabla f_i(x^*)\| = 0$. In other words, $\nabla f_i(x') = \nabla f_i(x^*)$, and thus $\mathbb{V}\left[\nabla f_i(x^*)\right] = \mathbb{V}\left[\nabla f_i(x')\right]$.

3. If interpolation holds, then there exists (see Lemma 4.10) $x \in \operatorname{argmin} f$ such that $x^* \in \operatorname{argmin} f_i$ for every $i = 1, \ldots, n$. From Fermat's theorem, this implies that $\nabla f_i(x^*) = 0$ and $\nabla f(x^*) = 0$. Consequently $\mathbb{V}[\nabla f_i(x^*)] = \mathbb{E}[||\nabla f_i(x^*) - \nabla f(x^*)||^2] = 0$. This proves that $\sigma_f^* = 0$. Now, if Assumption (Sum of Convex) holds and $\sigma_f^* = 0$, then we can use the previous item to say that for any $x^* \in \operatorname{argmin} f$ we have $\mathbb{V}[\nabla f_i(x^*)] = 0$. By definition of the variance and the fact that $\nabla f(x^*) = 0$, this implies that for every $i = 1, \ldots, n, \nabla f_i(x^*) = 0$. Using again the convexity of the f_i 's, we deduce that $x^* \in \operatorname{argmin} f_i$, which means that interpolation holds.

Both σ_f^* and Δ_f^* measure how far we are from interpolation. Furthermore, these two constants are related through the following bounds.

Lemma 4.18. Let Assumption (Sum of L_{max} -Smooth) hold.

- 1. We have $\sigma_f^* \leq 2L_{\max}\Delta_f^*$.
- 2. If moreover each f_i is μ -strongly convex, then $2\mu\Delta_f^* \leq \sigma_f^*$.

Proof.

- 1. Let $x^* \in \operatorname{argmin} f$. Using Lemma 2.28, we can write $\|\nabla f_i(x^*)\|^2 \leq 2L_{\max}(f_i(x^*) \inf f_i)$ for each *i*. The conclusion follows directly after taking the expectation on this inequality, and using the fact that $\mathbb{E}\left[\|\nabla f_i(x^*)\|^2\right] = \mathbb{V}\left[\nabla f_i(x^*)\right] \geq \sigma_f^*$.
- 2. This is exactly the same proof, except that we use Lemma 2.18 instead of 2.28.

4.3.3 Variance transfer

Here we provide two lemmas which allow to exchange variance-like terms like $\mathbb{E}\left[\|\nabla f_i(x)\|^2\right]$ with interpolation constants and function values. This is important since $\mathbb{E}\left[\|\nabla f_i(x)\|^2\right]$ actually controls the variance of the gradients (see Lemma 4.6).

Lemma 4.19 (Variance transfer : function noise). If Assumption (Sum of L_{max} -Smooth) holds, then for all $x \in \mathbb{R}^d$ we have

$$\mathbb{E}\left[\|\nabla f_i(x)\|^2\right] \le 2L_{\max}(f(x) - \inf f) + 2L_{\max}\Delta_f^*.$$

Proof. Let $x \in \mathbb{R}^d$ and $x^* \in \operatorname{argmin} f$. Using Lemma 2.28, we can write

$$\|\nabla f_i(x)\|^2 \le 2L_{\max}(f_i(x) - \inf f_i) = 2L_{\max}(f_i(x) - f_i(x^*)) + 2L_{\max}(f_i(x^*) - \inf f_i), \quad (37)$$

for each i. The conclusion follows directly after taking expectation over the above inequality. \Box

Lemma 4.20 (Variance transfer : gradient noise). If Assumptions (Sum of L_{max} -Smooth) and (Sum of Convex) hold, then for all $x \in \mathbb{R}^d$ we have that

$$\mathbb{E}\left[\|\nabla f_i(x)\|^2\right] \le 4L_{\max}(f(x) - \inf f) + 2\sigma_f^*.$$
(38)

Proof. Let $x^* \in \operatorname{argmin} f$, so that $\sigma_f^* = \mathbb{V}[\|\nabla f_i(x^*)\|^2]$ according to Lemma 4.17. Start by writing

$$\|\nabla f_i(x)\|^2 \le 2\|\nabla f_i(x) - \nabla f_i(x^*)\|^2 + 2\|\nabla f_i(x^*)\|^2.$$

Taking the expectation over the above inequality, then applying Lemma 4.8 gives the result. \Box

5 Stochastic Gradient Descent

Algorithm 5.1 (SGD). Consider Problem (Sum of Functions). Let $x^0 \in \mathbb{R}^d$, and let $\gamma_t > 0$ be a sequence of step sizes. The Stochastic Gradient Descent (SGD) algorithm is given by the iterates $(x^t)_{t\in\mathbb{N}}$ where

$$i_t \in \{1, \dots n\}$$
 Sampled with probability $\frac{1}{n}$ (39)

$$x^{t+1} = x^t - \gamma_t \nabla f_{i_t}(x^t).$$
(40)

Remark 5.2 (Unbiased estimator of the gradient). An important feature of the (SGD) Algorithm is that at each iteration we follow the direction $-\nabla f_{i_t}(x^t)$, which is an *unbiased* estimator of $-\nabla f(x^t)$. Indeed, since

$$\mathbb{E}\left[\nabla f_i(x^t) \mid x^t\right] = \sum_{i=1}^n \frac{1}{n} \nabla f_i(x^t) = \nabla f(x^t).$$
(41)

5.1 Convergence for convex and smooth functions

The behaviour of the (SGD) algorithm is very dependant of the choice of the sequence of stepsizes γ_t . In our next Theorem 5.3 we prove the convergence of SGD with a sequence of stepsizes where

each γ_t is less than $\frac{1}{2L_{\text{max}}}$. The particular cases of constant stepsizes and of decreasing stepsizes are dealt with in Theorem 5.5.

Theorem 5.3. Let Assumptions (Sum of L_{\max} -Smooth) and (Sum of Convex) hold. Consider $(x^t)_{t\in\mathbb{N}}$ a sequence generated by the (SGD) algorithm, with a stepsize satisfying $0 < \gamma_t < \frac{1}{2L_{\max}}$. It follows that

$$\mathbb{E}\left[f(\bar{x}^t) - \inf f\right] \le \frac{\|x^0 - x^*\|^2}{2\sum_{k=0}^{t-1} \gamma_k (1 - 2\gamma_k L_{\max})} + \frac{\sum_{k=0}^{t-1} \gamma_k^2}{\sum_{k=0}^{t-1} \gamma_k (1 - 2\gamma_k L_{\max})} \sigma_f^*,\tag{42}$$

where $\bar{x}^t \stackrel{\text{def}}{=} \sum_{k=0}^{t-1} p_{t,k} x^k$, with $p_{t,k} \stackrel{\text{def}}{=} \frac{\gamma_k (1-2\gamma_k L_{\max})}{\sum_{i=0}^{t-1} \gamma_i (1-2\gamma_i L_{\max})}$.

Proof. Let $x^* \in \operatorname{argmin} f$, so we have $\sigma_f^* = \mathbb{V}[\nabla f_i(x^*)]$ (see Lemma 4.17). We will note $\mathbb{E}_k[\cdot]$ instead of $\mathbb{E}_k[\cdot | x^k]$, for simplicity. Let us start by analyzing the behaviour of $||x^k - x^*||^2$. By developing the squares, we obtain

$$\|x^{k+1} - x^*\|^2 = \|x^k - x^*\|^2 - 2\gamma_k \langle \nabla f_i(x^k), x^k - x^* \rangle + \gamma_k^2 \|\nabla f_i(x^k)\|^2$$

Hence, after taking the expectation conditioned on x_k , we can use the convexity of f and the variance transfer lemma to obtain:

$$\mathbb{E}_{k} \left[\|x^{k+1} - x^{*}\|^{2} \right] = \|x^{k} - x^{*}\|^{2} + 2\gamma_{k} \langle \nabla f(x^{k}), x^{*} - x^{k} \rangle + \gamma_{k}^{2} \mathbb{E}_{k} \left[\|\nabla f_{i}(x_{k})\|^{2} \right]$$

$$\stackrel{(2)+(38)}{\leq} \|x^{k} - x^{*}\|^{2} + 2\gamma_{k} (2\gamma_{k}L_{\max} - 1)(f(x^{k}) - \inf f)) + 2\gamma_{k}^{2} \sigma_{f}^{*}.$$

Rearranging and taking expectation, we have

$$2\gamma_k(1 - 2\gamma_k L_{\max})\mathbb{E}\left[f(x^k) - \inf f\right] \le \mathbb{E}\left[\|x^k - x^*\|^2\right] - \mathbb{E}\left[\|x^{t+1} - x^*\|^2\right] + 2\gamma_k^2 \sigma_f^*.$$

Summing over k = 0, ..., t - 1 and using telescopic cancellation gives

$$2\sum_{k=0}^{t-1}\gamma_k(1-2\gamma_k L_{\max})\mathbb{E}\left[f(x^k) - \inf f\right] \le \|x^0 - x^*\|^2 - \mathbb{E}\left[\|x^t - x^*\|^2\right] + 2\sigma_f^*\sum_{k=0}^{t-1}\gamma_k^2.$$

Since $\mathbb{E}\left[\|x^t - x^*\|^2\right] \ge 0$, dividing both sides by $2\sum_{i=0}^{t-1} \gamma_i (1 - 2\gamma_i L_{\max})$ gives:

$$\sum_{k=0}^{t-1} \mathbb{E}\left[\frac{\gamma_k (1 - 2\gamma_k L_{\max})}{\sum_{i=0}^{t-1} \gamma_i (1 - 2\gamma_i L_{\max})} (f(x^k) - \inf f)\right] \le \frac{\|x^0 - x^*\|^2}{2\sum_{i=0}^{t-1} \gamma_i (1 - 2\gamma_i L_{\max})} + \frac{\sigma_f^* \sum_{k=0}^{t-1} \gamma_k^2}{\sum_{i=0}^{t-1} \gamma_i (1 - 2\gamma_i L_{\max})}.$$

Finally, define for $k = 0, \dots, t-1$

$$p_{t,k} \stackrel{\text{def}}{=} \frac{\gamma_k (1 - 2\gamma_k L_{\max})}{\sum_{i=0}^{t-1} \gamma_i (1 - 2\gamma_i L_{\max})}$$

and observe that $p_{t,k} \ge 0$ and $\sum_{k=0}^{t-1} p_{t,k} = 1$. This allows us to treat the $(p_{t,k})_{k=0}^{t-1}$ as if it was a probability vector. Indeed, using that f is convex together with Jensen's inequality gives

$$\mathbb{E}\left[f(\bar{x}^{t}) - f(x^{*})\right] \leq \sum_{k=0}^{t-1} \mathbb{E}\left[\frac{\gamma_{k}(1 - 2\gamma_{k}L_{\max})}{\sum_{i=0}^{t-1}\gamma_{i}(1 - 2\gamma_{i}L_{\max})}(f(x^{k}) - \inf f)\right]$$
$$\leq \frac{\|x^{0} - x^{*}\|^{2}}{2\sum_{i=0}^{t-1}\gamma_{i}(1 - 2\gamma_{i}L_{\max})} + \frac{\sigma_{f}^{*}\sum_{k=0}^{t-1}\gamma_{k}^{2}}{\sum_{i=0}^{t-1}\gamma_{i}(1 - 2\gamma_{i}L_{\max})}.$$

Remark 5.4 (On the choice of stepsizes for (SGD)). Looking at the bound obtained in Theorem 5.3, we see that the first thing we want is $\sum_{s=0}^{\infty} \gamma_s = +\infty$ so that the first term (a.k.a the bias term) vanishes. This can be achieved with constant stepsizes, or with stepsizes of the form $\frac{1}{t^{\alpha}}$ with $\alpha < 1$ (see Theorem 5.5 below). The second term (a.k.a the variance term) is less trivial to analyse.

- If interpolation holds (see Definition 4.9), then the variance term σ_f^* is zero. This means that the expected values converge to zero at a rate of the order $\frac{1}{\sum_{s=0}^{t-1} \gamma_s}$. For constant stepsizes this gives a $O\left(\frac{1}{t}\right)$ rate. For decreasing stepsizes $\gamma_t = \frac{1}{t^{\alpha}}$ this gives a $O\left(\frac{1}{t^{1-\alpha}}\right)$ rate. We see that the best among those rates is obtained when $\alpha = 0$ and the decay in the stepsize is slower. In other words when the stepsize is constant. Thus when interpolation holds the problem is so easy that the stochastic algorithm behaves like the deterministic one and enjoys a 1/t rate with constant stepsize, as in Theorem 3.4.
- If interpolation does not hold the expected values will be asymptotically controlled by

$$\frac{\sum_{s=0}^{t-1} \gamma_s^2}{\sum_{s=0}^{t-1} \gamma_s}.$$

We see that we want γ_s to decrease as slowly as possible (so that the denominator is big) but at the same time that γ_s vanishes as fast as possible (so that the numerator is small). So a trade-off must be found. For constant stepsizes, this term becomes a constant O(1), and thus (SPGD) does not converge for constant stepsizes. For decreasing stepsizes $\gamma_t = \frac{1}{t^{\alpha}}$, this term becomes (omitting logarithmic terms) $O\left(\frac{1}{t^{\alpha}}\right)$ if $0 < \alpha \leq \frac{1}{2}$, and $O\left(\frac{1}{t^{1-\alpha}}\right)$ if $\frac{1}{2} \leq \alpha < 1$. So the best compromise for this bound is to take $\alpha = 1/2$. This case is detailed in the next Theorem.

Theorem 5.5. Let Assumptions (Sum of L_{\max} -Smooth) and (Sum of Convex) hold. Consider $(x^t)_{t\in\mathbb{N}}$ a sequence generated by the (SGD) algorithm with a stepsize $\gamma_t > 0$.

1. If $\gamma_t = \gamma < \frac{1}{2L_{\text{max}}}$, then for every $t \ge 1$ we have that

$$\mathbb{E}\left[f(\bar{x}^t) - f(x^*)\right] \le \frac{\|x^0 - x^*\|^2}{2\gamma(1 - 2\gamma L_{\max})} \frac{1}{t} + \frac{\gamma}{1 - 2\gamma L_{\max}} \sigma_f^*,\tag{43}$$

where $\bar{x}^t \stackrel{\text{def}}{=} \frac{1}{t} \sum_{k=0}^{t-1} x^k$.

2. If $\gamma_t = \frac{\gamma}{\sqrt{t+1}}$ with $\gamma \leq \frac{1}{2L_{\max}}$, then for every t we have that

$$\mathbb{E}\left[f(\bar{x}^k) - f(x^*)\right] \le \frac{\|x^0 - x^*\|^2}{2\gamma\sqrt{k}} + \frac{\gamma^2 \log(k)}{\gamma\sqrt{k}}\sigma_f^* = \mathcal{O}\left(\frac{\log(k)}{\sqrt{k}}\right).$$
(44)

where $\bar{x}^t \stackrel{\text{def}}{=} \sum_{k=0}^{t-1} p_{t,k} x^k$, with $p_{t,k} \stackrel{\text{def}}{=} \frac{\gamma_k (1-2\gamma_k L_{\max})}{\sum_{i=0}^{t-1} \gamma_i (1-2\gamma_i L_{\max})}$.

Proof. For the different choices of step sizes:

- 1. For $\gamma_t = \gamma$, it suffices to replace γ_t in (42).
- 2. For $\gamma_t = \frac{\gamma}{\sqrt{t+1}}$, we first need some estimates on the sum (of squares) of the stepsizes. Using an integral bound, we have that

$$\sum_{t=0}^{k-1} \gamma_t^2 = \gamma^2 \sum_{t=0}^{k-1} \frac{1}{t+1} \le \gamma^2 \int_{t=0}^{k-1} \frac{1}{t+1} = \gamma^2 \log(k).$$
(45)

Furthermore using the integral bound again we have that

$$\sum_{t=0}^{k-1} \gamma_t \geq \int_{t=1}^{k-1} \frac{\gamma}{\sqrt{t+1}} = 2\gamma \left(\sqrt{k} - \sqrt{2}\right).$$

$$(46)$$

Now using (45) and (46), together with the fact that $\gamma L_{\text{max}} \leq \frac{1}{2}$, we have that for $k \geq 2$

$$\sum_{i=0}^{k-1} \gamma_i (1 - 2\gamma_i L_{\max}) \ge 2\gamma \left(\sqrt{k} - \sqrt{2} - \gamma L_{\max} \log(k)\right) \ge 2\gamma \left(\sqrt{k} - \sqrt{2} - \log(\sqrt{k})\right).$$

Because $X \mapsto \frac{X}{2} - \ln(X)$ is increasing for $X \ge 2$, we know that $\frac{X}{2} - \ln(X) \ge \sqrt{2}$ for X large enough (for instance $X \ge 7$). So by taking $X = \sqrt{k}$, we deduce from the above inequality that for $k \ge 7^2$ we have

$$\sum_{i=0}^{k-1} \gamma_i (1 - 2\gamma_i L_{\max}) \ge \gamma \sqrt{k}$$

It remains to use the above inequality in (42) and (45) again to arrive at

$$\mathbb{E}\left[f(\bar{x}^{k}) - f(x^{*})\right] \leq \frac{\|x^{0} - x^{*}\|^{2}}{2\sum_{i=0}^{k-1}\gamma_{i}(1 - 2\gamma_{i}L_{\max})} + \frac{\sum_{t=0}^{k-1}\gamma_{t}^{2}}{\sum_{i=0}^{k-1}\gamma_{i}(1 - 2\gamma_{i}L_{\max})}\sigma_{f}^{*}$$
$$\leq \frac{\|x^{0} - x^{*}\|^{2}}{2\gamma\sqrt{k}} + \frac{\gamma^{2}\log(k)}{\gamma\sqrt{k}}\sigma_{f}^{*}$$
$$= \mathcal{O}\left(\frac{\log(k)}{\sqrt{k}}\right).$$

Corollary 5.6 ($\mathcal{O}(1/\epsilon^2)$ Complexity). Consider the setting of Theorem 5.5. Let $\epsilon \geq 0$ and let $\gamma = \frac{1}{2L_{\max}} \frac{1}{\sqrt{t}}$. It follows that for $t \geq 4$ that

$$t \ge \frac{1}{\epsilon^2} \left(2L_{\max} \|x^0 - x^*\|^2 + \frac{\sigma_f^*}{L_{\max}} \right)^2 \implies \mathbb{E} \left[f(\bar{x}^t) - f(x^*) \right] \le \epsilon$$

$$\tag{47}$$

Proof. From (43) plugging $\gamma = \frac{1}{2L_{\max}\sqrt{t}}$ we have

$$\mathbb{E}\left[f(\bar{x}^t) - f(x^*)\right] \le \frac{2L_{\max}\sqrt{t}\|x^0 - x^*\|^2}{2(1 - \frac{1}{\sqrt{t}})}\frac{1}{t} + \frac{1}{2L_{\max}\sqrt{t}}\frac{1}{1 - \frac{1}{\sqrt{t}}}\sigma_f^*.$$

Now note that for $t \ge 4$ we have that

$$\frac{1}{1 - \frac{1}{\sqrt{t}}} \le 2$$

Using this in the above we have that

$$\mathbb{E}\left[f(\bar{x}^{t}) - f(x^{*})\right] \leq \frac{1}{\sqrt{t}} \left(2L_{\max} \|x^{0} - x^{*}\|^{2} + \frac{\sigma_{f}^{*}}{L_{\max}}\right).$$

Consequently (47) follows by demanding that the above inequality is less than ϵ .

5.2 Convergence for strongly convex and smooth functions

Theorem 5.7. Let Assumptions (Sum of L_{\max} -Smooth) and (Sum of Convex) hold, and assume further that f is μ -strongly convex. Consider the $(x^t)_{t\in\mathbb{N}}$ sequence generated by the (SGD) algorithm with a constant stepsize satisfying $0 < \gamma < \frac{1}{2L_{\max}}$. It follows that for $t \ge 0$,

$$\mathbb{E}\|x^{t} - x^{*}\|^{2} \leq (1 - \gamma\mu)^{t} \|x^{0} - x^{*}\|^{2} + \frac{2\gamma}{\mu}\sigma_{f}^{*}.$$

Proof. Let $x^* \in \operatorname{argmin} f$, so that $\sigma_f^* = \mathbb{V}[\nabla f_i(x^*)]$ (see Lemma 4.17). We will note $\mathbb{E}_k[\cdot]$ instead of $\mathbb{E}[\cdot | x^k]$, for simplicity. Expanding the squares we have

$$||x^{t+1} - x^*||^2 \stackrel{(39)}{=} ||x^k - x^* - \gamma \nabla f_i(x^k)||^2$$

= $||x^k - x^*||^2 - 2\gamma \langle x^k - x^*, \nabla f_i(x^k) \rangle + \gamma^2 ||\nabla f_i(x^k)||^2.$

Taking expectation conditioned on x^k we obtain

$$\mathbb{E}_{k} \left[\|x^{t+1} - x^{*}\|^{2} \right] \stackrel{(41)}{=} \|x^{k} - x^{*}\|^{2} - 2\gamma \langle x^{k} - x^{*}, \nabla f(x^{k}) \rangle + \gamma^{2} \mathbb{E}_{k} \left[\|\nabla f_{i}(x^{k})\|^{2} \right] \\
\stackrel{Lem. 2.14}{\leq} (1 - \gamma \mu) \|x^{k} - x^{*}\|^{2} - 2\gamma [f(x^{k}) - f(x^{*})] + \gamma^{2} \mathbb{E}_{k} \left[\|\nabla f_{i}(x^{k})\|^{2} \right].$$

Taking expectations again and using Lemma 4.20 gives

$$\mathbb{E}\left[\|x^{t+1} - x^*\|^2\right] \stackrel{(38)}{\leq} (1 - \gamma\mu)\mathbb{E}\|x^k - x^*\|^2 + 2\gamma^2 \sigma_f^* + 2\gamma(2\gamma L_{\max} - 1)\mathbb{E}[f(x^k) - f(x^*)] \\ \leq (1 - \gamma\mu)\mathbb{E}\left[\|x^k - x^*\|^2\right] + 2\gamma^2 \sigma_f^*,$$

where we used in the last inequality that $2\gamma L_{\max} \leq 1$ since $\gamma \leq \frac{1}{2L_{\max}}$. Recursively applying the above and summing up the resulting geometric series gives

$$\mathbb{E}\|x^{k} - x^{*}\|^{2} \leq (1 - \gamma\mu)^{k} \|x^{0} - x^{*}\|^{2} + 2\sum_{j=0}^{k-1} (1 - \gamma\mu)^{j} \gamma^{2} \sigma_{f}^{*}$$
$$\leq (1 - \gamma\mu)^{k} \|x^{0} - x^{*}\|^{2} + \frac{2\gamma\sigma_{f}^{*}}{\mu}.$$

Corollary 5.8 ($\mathcal{O}(1/\epsilon)$ Complexity). Consider the setting of Theorem 5.7. Let $\epsilon > 0$. If we set

$$\gamma = \min\left\{\frac{\epsilon}{2}\frac{\mu}{2\sigma_f^*}, \frac{1}{2L_{\max}}\right\}$$
(48)

then

$$t \ge \max\left\{\frac{1}{\epsilon}\frac{4\sigma_f^*}{\mu^2}, \ \frac{2L_{\max}}{\mu}\right\}\log\left(\frac{2\|x^0 - x^*\|^2}{\epsilon}\right) \implies \|x^t - x^*\|^2 \le \epsilon \tag{49}$$

Proof. Applying Lemma A.2 with $A = \frac{2\sigma_f^*}{\mu}$, $C = 2L_{\text{max}}$ and $\alpha_0 = \|x^0 - x^*\|^2$ gives the result (49).

5.3 Convergence for Polyak-Lojasiewicz and smooth functions

Theorem 5.9. Let Assumption (Sum of L_{\max} -Smooth) hold, and assume that f is μ -Polyak-Lojasiewicz for some $\mu > 0$. Consider $(x^t)_{t \in \mathbb{N}}$ a sequence generated by the (SGD) algorithm, with a constant stepsize satisfying $0 < \gamma \leq \frac{\mu}{L_f L_{\max}}$. It follows that

$$\mathbb{E}[f(x^t) - \inf f] \le (1 - \gamma \mu)^t (f(x^0) - \inf f) + \frac{\gamma L_f L_{\max}}{\mu} \Delta_f^*.$$

Proof. Remember from Assumption (Sum of L_{max} -Smooth) that f is L_f -smooth, so we can use Lemma 2.25, together with the update rule of SGD, to obtain:

$$\begin{aligned} f(x^{t+1}) &\leq f(x^{t}) + \langle \nabla f(x^{t}), x^{t+1} - x^{t} \rangle + \frac{L_{f}}{2} \|x^{t+1} - x^{t}\|^{2} \\ &= f(x^{t}) - \gamma \langle \nabla f(x^{t}), \nabla f_{i}(x^{t}) \rangle + \frac{L_{f} \gamma^{2}}{2} \|\nabla f_{i}(x^{t})\|^{2}. \end{aligned}$$

After taking expectation conditioned on x^t , we can use a variance transfer lemma together with the Polyak-Lojasiewicz property to write

$$\mathbb{E}\left[f(x^{t+1}) \mid x^{t}\right] \leq f(x^{t}) - \gamma \|\nabla f(x^{t})\|^{2} + \frac{L_{f}\gamma^{2}}{2} \mathbb{E}\left[\|\nabla f_{i}(x^{t})\|^{2} \mid x^{t}\right] \\ \stackrel{\text{Lemma 4.19}}{\leq} f(x^{t}) - \gamma \|\nabla f(x^{t})\|^{2} + \gamma^{2}L_{f}L_{\max}(f(x^{t}) - \inf f) + \gamma^{2}L_{f}L_{\max}\Delta_{f}^{*} \\ \stackrel{\text{Definition 2.17}}{\leq} f(x^{t}) + \gamma(\gamma L_{f}L_{\max} - 2\mu)(f(x^{t}) - \inf f) + \gamma^{2}L_{f}L_{\max}\Delta_{f}^{*} \\ \leq f(x^{t}) - \gamma\mu(f(x^{t}) - \inf f) + \gamma^{2}L_{f}L_{\max}\Delta_{f}^{*},$$

where in the last inequality we used our assumption on the stepsize to write $\gamma L_f L_{\text{max}} - 2\mu \leq -\mu$. Note that $\mu \gamma \leq 1$ because of our assumption on the stepsize, and the fact that $\mu \leq L_f \leq L_{\text{max}}$ (see Remark 2.27). Subtracting inf f from both sides in the last inequality, and taking expectation, we obtain

$$\mathbb{E}\left[f(x^{t+1}) - \inf f\right] \leq (1 - \mu\gamma)\mathbb{E}\left[f(x^t) - \inf f\right] + \gamma^2 L_f L_{\max}\Delta_f^*.$$

Recursively applying the above and summing up the resulting geometric series gives:

$$\mathbb{E}\left[f(x^t) - \inf f\right] \leq (1 - \mu\gamma)^t \mathbb{E}\left[f(x^0) - \inf f\right] + \gamma^2 L_f L_{\max} \Delta_f^* \sum_{j=0}^{t-1} (1 - \gamma\mu)^j.$$

Using $\sum_{i=0}^{t-1} (1 - \mu \gamma)^i = \frac{1 - (1 - \mu \gamma)^t}{1 - 1 + \mu \gamma} \le \frac{1}{\mu \gamma}$, in the above gives (5.9).

Corollary 5.10 ($\tilde{\mathcal{O}}(1/\epsilon)$ Complexity). Consider the setting of Theorem 5.9. Let $\epsilon \geq 0$ be given. If we set

$$\gamma = \frac{\mu}{L_f L_{\max}} \min\left\{\frac{\epsilon}{2\Delta_f^*}, 1\right\}$$
(50)

then

$$t \ge \frac{L_f L_{\max}}{\mu^2} \max\left\{\frac{2\Delta_f^*}{\epsilon}, 1\right\} \log\left(\frac{2(f(x^0) - \inf f)}{\epsilon}\right) \implies f(x^0) - \inf f \le \epsilon$$
(51)

Proof. Applying Lemma A.2 with $A = \frac{L_f L_{\text{max}}}{\mu} \Delta_f^*$ and $\alpha_0 = f(x^0) - \inf f$ gives the result (51).

5.4 Bibliographic notes

The early and foundational works on SGD include [35, 28, 29, 40, 30, 18], though these references are either for the non-smooth setting for Lipschitz losses, or are asymptotic. The first non-asymptotic analyses of SGD the smooth and convex setting that we are aware of is in [25], closely followed by [38] under a different growth assumption. These results were later improved in [26], where the authors removed the quadratic dependency on the smoothness constant and considered importance sampling. The proof of Theorem 5.3 is a simplified version of [16, Theorem D.6]. The proof of Theorem 5.7 is a simplified version of [17, Theorem 3.1]. The proof of Theorem 5.9 has been adapted from the proof of [16, Theorem 4.6].

For a general convergence theory for SGD in the smooth and non-convex setting we recommend [20]. Also, the definition of function noise that we use here was taken from [20]. The first time we saw Lemma 4.19 was also in [20]. Theorem 5.9, which relies on the Polyak-Lojasiewicz condition, is based on the proof in [16], with the only different being that we use function noise as opposed to gradient noise. This Theorem 5.9 is also very similar to Theorem 3 in [20], with the difference being that Theorem 3 in [20] is more general (uses weaker assumptions), but also has a more involved proof and a different step size.

An excellent reference for proof techniques for SGD focused on the online setting is the recent book [32], which contains proofs for adaptive step sizes such a Adagrad and coin tossing based step sizes.

Remark 5.11 (From finite sum to expectation). The theorems we prove here also holds when

the objective is a true expectation where

$$f(x) = \mathbb{E}_{\xi \sim \mathcal{D}} \left[f(x, \xi) \right].$$

Further we have defined the L_{max} smoothness as the largest smoothness constant of every $f(x,\xi)$ for every ξ . The gradient noise σ_f^* is would now be given by

$$\sigma_f^* \stackrel{\text{def}}{=} \inf_{x^* \in \operatorname{argmin} f} \mathbb{E}_{\xi} \left[\|\nabla f(x^*, \xi)\|^2 \right].$$

The function noise would now be given by

$$\Delta_f^* \stackrel{\text{def}}{=} \inf_{x \in \mathbb{R}^d} f - \mathbb{E}_{\xi} \left[\inf_{x \in \mathbb{R}^d} f(x,\xi) \right]$$

With these extended definitions we have that Theorems 5.5, 5.7 and 5.9 hold verbatim.

We also give some results for minimizing expectation for stochastic subgradient in Section 9.

6 Minibatch SGD

6.1 Definitions

When solving (Sum of Functions) in practice, an estimator of the gradient is often computed using a small batch of functions, instead of a single one as in (SGD). More precisely, given a subset $B \subset \{1, \ldots, n\}$, we want to make use of

$$\nabla f_B(x^t) \stackrel{\text{def}}{=} \frac{1}{|B|} \sum_{i \in B} \nabla f_i(x^t)$$

This leads to the *minibatching* SGD algorithm:

Algorithm 6.1 (MiniSGD). Let $x^0 \in \mathbb{R}^d$, let a batch size $b \in \{1, \ldots, n\}$, and let $\gamma_t > 0$ be a sequence of step sizes. The Minibatching Stochastic Gradient Descent (MiniSGD) algorithm is given by the iterates $(x^t)_{t \in \mathbb{N}}$ where

$$B_t \subset \{1, \dots n\}$$
 Sampled uniformly among sets of size b
$$x^{t+1} = x^t - \gamma_t \nabla f_{B_t}(x^t).$$
 (52)

Remark 6.2 (On random batches). We impose in this section that the batches B are sampled uniformly among all subsets of size b in $\{1, \ldots, n\}$. This means that each batch is sampled with probability

$$\frac{1}{\binom{n}{b}} = \frac{(n-b)!b!}{n!}$$

and that we will compute expectation and variance with respect to this uniform law. For instance

the expectation of the minibatched gradient writes as

$$\mathbb{E}_b\left[\nabla f_B(x)\right] = \frac{1}{\binom{n}{b}} \sum_{\substack{B \subset \{1,\dots,n\}\\|B|=b}} \nabla f_B(x),$$

and it is an exercise to verify that this is exactly equal to $\nabla f(x)$.

Mini-batching makes better use of parallel computational resources and it also speeds-up the convergence of (SGD), as we show next. To do so, we will need the same central tools than for (SGD), that is the notions of gradient noise, of expected smoothness, and a variance transfer lemma.

Definition 6.3. Let Assumption (Sum of L_{max} -Smooth) hold, and let $b \in \{1, \ldots, n\}$. We define the **minbatch gradient noise** as

$$\sigma_b^* \stackrel{\text{def}}{=} \inf_{x^* \in \operatorname{argmin} f} \, \mathbb{V}_b \left[\nabla f_B(x^*) \right], \tag{53}$$

where the variance is taken over the random variable B, sampled uniformly among the subsets of size b in $\{1, \ldots, n\}$.

Definition 6.4. Let Assumption (Sum of L_{max} -Smooth) hold, and let $b \in \{1, \ldots, n\}$. We say that f is \mathcal{L}_b -smooth in expectation if

for all
$$x, y \in \mathbb{R}^d$$
, $\frac{1}{2\mathcal{L}_b} \mathbb{E}_b \left[\|\nabla f_i(y) - \nabla f_i(x)\|^2 \right] \leq f(y) - f(x) - \langle \nabla f(x), y - x \rangle$.

Lemma 6.5 (From single batch to minibatch). Let Assumptions (Sum of L_{max} -Smooth) and (Sum of Convex) hold. Then f is \mathcal{L}_b -smooth in expectation with

$$\mathcal{L}_{b} = \frac{n(b-1)}{b(n-1)}L + \frac{n-b}{b(n-1)}L_{\max},$$
(54)

and the minibatch gradient noise can be computed via

$$\sigma_b^* = \frac{n-b}{b(n-1)}\sigma_f^*.$$
(55)

Remark 6.6 (Minibatch interpolates between single and full batches). It is intersting to look at variations of the expected smoothness constant \mathcal{L}_b and minibatch gradient noise σ_b^* when bvaries from 1 to n. For b = 1, where (MiniSGD) reduces to (SGD), we have that $\mathcal{L}_b = L_{\max}$ and $\sigma_b^* = \sigma_f^*$, which are the constants governing the complexity of (SGD) as can be seen in Section 5. On the other extreme, when b = n (MiniSGD) reduces to (GD), we see that $\mathcal{L}_b = L$ and $\sigma_b^* = 0$. We recover the fact that the behavior of (GD) is controlled by the Lipschitz constant L, and has no variance.

We end this presentation with a variance transfer lemma, analog to Lemma 4.20 (resp. Lemma 2.29) in the single batch (resp. full batch).

Lemma 6.7. Let Assumptions (Sum of L_{max} -Smooth) and (Sum of Convex) hold. It follows that

$$\mathbb{E}_b\left[\|\nabla f_B(x)\|^2\right] \le 4\mathcal{L}_b(f(x) - \inf f) + 2\sigma_b^*$$

Proof of Lemmas 6.5 and 6.7. See Proposition 3.8 and 3.10 in [17].

6.2 Convergence for convex and smooth functions

Theorem 6.8. Let Assumptions (Sum of L_{\max} -Smooth) and (Sum of Convex) hold. Consider $(x^t)_{t\in\mathbb{N}}$ a sequence generated by the (MiniSGD) algorithm, with a sequence of stepsizes satisfying $0 < \gamma_t < \frac{1}{2\mathcal{L}_b}$. It follows that

$$\mathbb{E}\left[f(\bar{x}^{t}) - \inf f\right] \leq \frac{\|x^{0} - x^{*}\|^{2}}{2\sum_{k=0}^{t-1} \gamma_{k}(1 - 2\gamma_{k}\mathcal{L}_{b})} + \frac{\sum_{k=0}^{t-1} \gamma_{k}^{2}}{\sum_{k=0}^{t-1} \gamma_{k}(1 - 2\gamma_{k}\mathcal{L}_{b})}\sigma_{b}^{*}$$

where $\bar{x}^t \stackrel{\text{def}}{=} \sum_{k=0}^{t-1} p_{t,k} x^k$, with $p_{t,k} \stackrel{\text{def}}{=} \frac{\gamma_k (1-2\gamma_k \mathcal{L}_b)}{\sum_{i=0}^{t-1} \gamma_i (1-2\gamma_i \mathcal{L}_b)}$.

Proof. Let $x^* \in \operatorname{argmin} f$, so we have $\sigma_b^* = \mathbb{V}[\nabla f_B(x^*)]$. Let us start by analyzing the behaviour of $||x^k - x^*||^2$. By developing the squares, we obtain

$$\|x^{k+1} - x^*\|^2 = \|x^k - x^*\|^2 - 2\gamma_k \langle \nabla f_{B_t}(x^k), x^k - x^* \rangle + \gamma_k^2 \|\nabla f_{B_t}(x^k)\|^2$$

Hence, after taking the expectation conditioned on x_k , we can use the convexity of f and the variance transfer lemma to obtain:

$$\mathbb{E}_{b}\left[\|x^{k+1} - x^{*}\|^{2} \mid x^{k}\right] = \|x^{k} - x^{*}\|^{2} + 2\gamma_{k}\langle \nabla f(x^{k}), x^{*} - x^{k}\rangle + \gamma_{k}^{2}\mathbb{E}_{b}\left[\|\nabla f_{B_{t}}(x_{k})\|^{2} \mid x^{k}\right]$$

$$\stackrel{Lem. 2.8 \& 6.7}{\leq} \|x^{k} - x^{*}\|^{2} + 2\gamma_{k}(2\gamma_{k}\mathcal{L}_{b} - 1)(f(x^{k}) - \inf f)) + 2\gamma_{k}^{2}\sigma_{b}^{*}.$$

Rearranging and taking expectation, we have

$$2\gamma_k(1 - 2\gamma_k\mathcal{L}_b)\mathbb{E}_b\left[f(x^k) - \inf f\right] \le \mathbb{E}_b\left[\|x^k - x^*\|^2\right] - \mathbb{E}_b\left[\|x^{t+1} - x^*\|^2\right] + 2\gamma_k^2\sigma_b^*.$$

Summing over k = 0, ..., t - 1 and using telescopic cancellation gives

$$2\sum_{k=0}^{t-1} \gamma_k (1 - 2\gamma_k \mathcal{L}_b) \mathbb{E}_b \left[f(x^k) - \inf f \right] \le \|x^0 - x^*\|^2 - \mathbb{E}_b \left[\|x^t - x^*\|^2 \right] + 2\sigma_b^* \sum_{k=0}^{t-1} \gamma_k^2.$$

Since $\mathbb{E}_b\left[\|x^t - x^*\|^2\right] \ge 0$, dividing both sides by $2\sum_{i=0}^{t-1} \gamma_i(1 - 2\gamma_i \mathcal{L}_b)$ gives:

$$\sum_{k=0}^{t-1} \mathbb{E}_b \left[\frac{\gamma_k (1 - 2\gamma_k \mathcal{L}_b)}{\sum_{i=0}^{t-1} \gamma_i (1 - 2\gamma_i \mathcal{L}_b)} (f(x^k) - \inf f) \right] \le \frac{\|x^0 - x^*\|^2}{2\sum_{i=0}^{t-1} \gamma_i (1 - 2\gamma_i \mathcal{L}_b)} + \frac{\sigma_b^* \sum_{k=0}^{t-1} \gamma_k^2}{\sum_{i=0}^{t-1} \gamma_i (1 - 2\gamma_i \mathcal{L}_b)}.$$

Finally, define for $k = 0, \ldots, t - 1$

$$p_{t,k} \stackrel{\text{def}}{=} \frac{\gamma_k (1 - 2\gamma_k \mathcal{L}_b)}{\sum_{i=0}^{t-1} \gamma_i (1 - 2\gamma_i \mathcal{L}_b)}$$

and observe that $p_{t,k} \ge 0$ and $\sum_{k=0}^{t-1} p_{t,k} = 1$. This allows us to treat the $(p_{t,k})_{k=0}^{t-1}$ as if it was a probability vector. Indeed, using that f is convex together with Jensen's inequality gives

$$\mathbb{E}_{b}\left[f(\bar{x}^{t}) - \inf f\right] \leq \sum_{k=0}^{t-1} \mathbb{E}_{b}\left[\frac{\gamma_{k}(1 - 2\gamma_{k}\mathcal{L}_{b})}{\sum_{i=0}^{t-1}\gamma_{i}(1 - 2\gamma_{i}\mathcal{L}_{b})}(f(x^{k}) - \inf f)\right] \\ \leq \frac{\|x^{0} - x^{*}\|^{2}}{2\sum_{i=0}^{t-1}\gamma_{i}(1 - 2\gamma_{i}\mathcal{L}_{b})} + \frac{\sigma_{b}^{*}\sum_{k=0}^{t-1}\gamma_{k}^{2}}{\sum_{i=0}^{t-1}\gamma_{i}(1 - 2\gamma_{i}\mathcal{L}_{b})}.$$

Theorem 6.9. Let Assumptions (Sum of L_{\max} -Smooth) and (Sum of Convex) hold. Consider $(x^t)_{t\in\mathbb{N}}$ a sequence generated by the (MiniSGD) algorithm with a sequence of stepsizes $\gamma_t > 0$.

1. If $\gamma_t = \gamma < \frac{1}{2\mathcal{L}_b}$, then for every $t \ge 1$ we have that

$$\mathbb{E}\left[f(\bar{x}^t) - \inf f\right] \le \frac{\|x^0 - x^*\|^2}{2\gamma(1 - 2\gamma\mathcal{L}_b)} \frac{1}{t} + \frac{\gamma}{1 - 2\gamma\mathcal{L}_b} \sigma_b^*,$$

where $\bar{x}^t \stackrel{\text{def}}{=} \frac{1}{t} \sum_{k=0}^{t-1} x^k$.

2. If $\gamma_t = \frac{\gamma}{\sqrt{t+1}}$ with $\gamma \leq \frac{1}{2\mathcal{L}_b}$, then for every t we have that

$$\mathbb{E}\left[f(\bar{x}^t) - \inf f\right] \leq \frac{\|x^0 - x^*\|^2}{2\gamma\sqrt{t}} + \frac{\gamma^2\log(t)}{\gamma\sqrt{t}}\sigma_b^* = \mathcal{O}\left(\frac{\log(t)}{\sqrt{t}}\right).$$

where $\bar{x}^t \stackrel{\text{def}}{=} \sum_{k=0}^{t-1} p_{t,k}x^k$, with $p_{t,k} \stackrel{\text{def}}{=} \frac{\gamma_k(1-2\gamma_k\mathcal{L}_b)}{\sum_{i=0}^{t-1}\gamma_i(1-2\gamma_i\mathcal{L}_b)}.$

Corollary 6.10 ($\mathcal{O}(1/\epsilon^2)$ Complexity). Consider the setting of Theorem 6.8. Let $\epsilon \geq 0$ and let $\gamma = \frac{1}{2\mathcal{L}_b} \frac{1}{\sqrt{t}}$. It follows that for $t \geq 4$ that

$$t \ge \frac{1}{\epsilon^2} \left(2\mathcal{L}_b \| x^0 - x^* \|^2 + \frac{\sigma_b^*}{\mathcal{L}_b} \right)^2 \implies \mathbb{E} \left[f(\bar{x}^t) - \inf f \right] \le \epsilon.$$

Proof. The proof is a consequence of Theorem 6.8 and follows exactly the lines of the proofs of Theorem 5.5 and Corollary 5.6.

6.3 Rates for strongly convex and smooth functions

Theorem 6.11. Let Assumptions (Sum of L_{\max} -Smooth) and (Sum of Convex) hold, and assume further that f is μ -strongly convex. Consider $(x^t)_{t\in\mathbb{N}}$ a sequence generated by the (MiniSGD) algorithm, with a constant sequence of stepsizes $\gamma_t \equiv \gamma \in]0, \frac{1}{2\mathcal{L}_b}]$. Then

$$\mathbb{E}_{b}\left[\|x^{t} - x^{*}\|^{2}\right] \leq (1 - \gamma\mu)^{t} \|x^{0} - x^{*}\|^{2} + \frac{2\gamma\sigma_{b}^{*}}{\mu}.$$

Proof. Let $x^* \in \operatorname{argmin} f$, so that $\sigma_b^* = \mathbb{V}_b[\nabla f_B(x^*)]$. Expanding the squares we have

$$\begin{aligned} \|x^{t+1} - x^*\|^2 &\stackrel{(MiniSGD)}{=} \|x^t - x^* - \gamma \nabla f_{B_t}(x^t)\|^2 \\ &= \|x^t - x^*\|^2 - 2\gamma \langle x^t - x^*, \nabla f_{B_t}(x^t) \rangle + \gamma^2 \|\nabla f_{B_t}(x^t)\|^2 \end{aligned}$$

Taking expectation conditioned on x^t and using $\mathbb{E}_b [\nabla f_B(x)] = \nabla f(x)$ (see Remark 6.2), we obtain

$$\mathbb{E}_{b} \left[\|x^{t+1} - x^{*}\|^{2} \mid x^{t} \right] = \|x^{t} - x^{*}\|^{2} - 2\gamma \langle x^{t} - x^{*}, \nabla f(x^{t}) \rangle + \gamma^{2} \mathbb{E}_{b} \left[\|\nabla f_{B_{t}}(x^{t})\|^{2} \mid x^{t} \right] \\
\stackrel{Lem.2.14}{\leq} (1 - \gamma \mu) \|x^{t} - x^{*}\|^{2} - 2\gamma [f(x^{t}) - \inf f] + \gamma^{2} \mathbb{E}_{b} \left[\|\nabla f_{B_{t}}(x^{t})\|^{2} \mid x^{t} \right].$$

Taking expectations again and using Lemma 6.7 gives

$$\mathbb{E}_{b} \left[\|x^{t+1} - x^{*}\|^{2} \right] \leq (1 - \gamma \mu) \mathbb{E}_{b} \left[\|x^{t} - x^{*}\|^{2} \right] + 2\gamma^{2} \sigma_{b}^{*} + 2\gamma (2\gamma \mathcal{L}_{b} - 1) \mathbb{E}[f(x^{t}) - \inf f] \\ \leq (1 - \gamma \mu) \mathbb{E}_{b} \left[\|x^{t} - x^{*}\|^{2} \right] + 2\gamma^{2} \sigma_{b}^{*},$$

where we used in the last inequality that $2\gamma \mathcal{L}_b \leq 1$ since $\gamma \leq \frac{1}{2\mathcal{L}_b}$. Recursively applying the above and summing up the resulting geometric series gives

$$\mathbb{E}_{b} \left[\|x^{t} - x^{*}\|^{2} \right] \leq (1 - \gamma \mu)^{t} \|x^{0} - x^{*}\|^{2} + 2 \sum_{k=0}^{t-1} (1 - \gamma \mu)^{k} \gamma^{2} \sigma_{b}^{*}$$

$$\leq (1 - \gamma \mu)^{t} \|x^{0} - x^{*}\|^{2} + \frac{2\gamma \sigma_{b}^{*}}{\mu}.$$

Corollary 6.12 ($\tilde{\mathcal{O}}(1/\epsilon)$ Complexity). Consider the setting of Theorem 6.11. Let $\epsilon > 0$ be given. Hence, given any $\epsilon > 0$, choosing stepsize

$$\gamma = \min\left\{\frac{1}{2\mathcal{L}_b}, \ \frac{\epsilon\mu}{4\sigma_b^*}\right\},\tag{56}$$

and

$$t \ge \max\left\{\frac{2\mathcal{L}_b}{\mu}, \frac{4\sigma_b^*}{\epsilon\mu^2}\right\} \log\left(\frac{2\|x^0 - x^*\|^2}{\epsilon}\right) \implies \mathbb{E}\|x^t - x^*\|^2 \le \epsilon.$$
(57)

Proof. Applying Lemma A.2 with $A = \frac{2\sigma_b^*}{\mu}$, $C = 2\mathcal{L}_b$ and $\alpha_0 = ||x^0 - x^*||^2$ gives the result (57).

6.4 Bibliographic Notes

The SGD analysis in [26] was later extended to a mini-batch analysis [27], but restricted to minibatches that are disjoint partitions of the data. Our results on mini-batching in Section 6 are instead taken from [17]. We choose to adapt the proofs from [17] since these proofs allow for sampling with replacement. The smoothness constant in (54) was introduced in [15] and this particular formula was conjectured in [11].

7 Stochastic Momentum

For most, if not all, machine learning applications SGD is used with *momentum*. In the machine learning community, the *momentum method* is often written as follows

Algorithm 7.1 (Momentum). Let Assumption (Sum of L_{\max} -Smooth) hold. Let $x^0 \in \mathbb{R}^d$ and $m^{-1} = 0$, let $(\gamma_t)_{t \in \mathbb{N}} \subset [0, +\infty)$ be a sequence of stepsizes, and let $(\beta_t)_{t \in \mathbb{N}} \subset [0, 1]$ be a sequence of momentum parameters. The **Momentum** algorithm defines a sequence $(x^t)_{t \in \mathbb{N}}$ satisfying for every $t \in \mathbb{N}$

$$m^{t} = \beta_{t} m^{t-1} + \nabla f_{i_{t}}(x^{t}),$$

$$x^{t+1} = x^{t} - \gamma_{t} m^{t}.$$
 (58)

At the end of this section we will see in Corollary 7.4 that in the convex setting, the sequence x^t generated by the (Momentum) algorithm has a complexity rate of $\mathcal{O}(1/\varepsilon^2)$. This is an improvement with respect to (SGD), for which we only know complexity results about the *average of the iterates*, see Corollary 5.6.

7.1 The many ways of writing momentum

In the optimization community the momentum method is often written in the *heavy ball* format which is

$$x^{t+1} = x^t - \hat{\gamma}_t \,\nabla f_{i_t}(x^t) + \hat{\beta}_t(x^t - x^{t-1}), \tag{59}$$

where $\hat{\beta}_t \in [0, 1]$ is another momentum parameter, $i_t \in \{1, \ldots, n\}$ is sampled uniformly and i.i.d at each iteration. These two ways of writing down momentum in (Momentum) and (59) are equivalent, as we show next.

Lemma 7.2. The algorithms (Momentum) and Heavy Ball (given by (59)) are the equivalent. More precisely, if $(x^t)_{t\in\mathbb{N}}$ is generated by (Momentum) from parameters γ_t , β_t , then it verifies (59) by taking $\hat{\gamma}_t = \gamma_t$ and $\hat{\beta}_t = \frac{\gamma_t \beta_t}{\gamma_{t-1}}$, where $\gamma_{-1} = 0$.

Proof. Starting from (Momentum) we have that

$$x^{t+1} = x^t - \gamma_t m^t$$

= $x^t - \gamma_t \beta_t m^{t-1} - \gamma_t \nabla f_i(x^t).$

Now using (58) at time t-1 we have that $m^{t-1} = \frac{x^{t-1}-x^t}{\gamma_{t-1}}$ which when inserted in the above gives

$$x^{t+1} = x^t - \frac{\gamma_t \beta_t}{\gamma_{t-1}} (x^{t-1} - x^t) - \gamma_t \nabla f_i(x^t).$$

The conclusion follows by taking $\hat{\gamma}_t = \gamma_t$ and $\hat{\beta}_t = \frac{\gamma_t \beta_t}{\gamma_{t-1}}$.

There is yet a third equivalent way of writing down the momentum method that will be useful in establishing convergence.

Lemma 7.3. The algorithm (Momentum) is equivalent to the following *iterate-moving-average* (IMA) algorithm : start from $z^{-1} = 0$ and iterate for $t \in \mathbb{N}$

$$z^{t} = z^{t-1} - \eta_t \nabla f_{i_t}(x^t), (60)$$

$$x^{t+1} = \frac{\lambda_{t+1}}{\lambda_{t+1}+1}x^t + \frac{1}{\lambda_{t+1}+1}z^t.$$
 (61)

More precisely, if $(x^t)_{t\in\mathbb{N}}$ is generated by (Momentum) from parameters γ_t, β_t , then it verifies (IMA) by chosing any parameters (η_t, λ_t) satisfying

$$\beta_t \lambda_{t+1} = \frac{\gamma_{t-1} \lambda_t}{\gamma_t} - \beta_t, \quad \eta_t = (1 + \lambda_{t+1}) \gamma_t, \quad \text{and} \quad z^{t-1} = x^t + \lambda_t (x^t - x^{t-1}).$$

Proof. Let $(x^t)_{t\in\mathbb{N}}$ be generated by (Momentum) from parameters γ_t, β_t . By definition of z^t , we have

$$z^{t} = (1 + \lambda_{t+1})x^{t+1} - \lambda_{t+1}x^{t},$$

which after dividing by $(1 + \lambda_{t+1})$ directly gives us (61). Now use Lemma 7.2 to write that $x^{t+1} = x^t - \gamma_t \nabla f_{i_t}(x^t) + \hat{\beta}_t(x^t - x^{t-1})$ where $\hat{\beta}_t = \frac{\gamma_t \beta_t}{\gamma_{t-1}}$. Going back to the definition of z^t , we can write

$$z^{t} = (1 + \lambda_{t+1})x^{t+1} - \lambda_{t+1}x^{t}$$

= $(1 + \lambda_{t+1})(x^{t} - \gamma_{t} \nabla f_{i_{t}}(x^{t}) + \hat{\beta}_{t}(x^{t} - x^{t-1})) - \lambda_{t+1}x^{t}$
= $x^{t} - (1 + \lambda_{t+1})\gamma_{t} \nabla f_{i_{t}}(x^{t}) + (1 + \lambda_{t+1})\hat{\beta}_{t}(x^{t} - x^{t-1}))$
= $z^{t-1} - \eta_{t} \nabla f_{i_{t}}(x^{t}),$

where in the last equality we used the fact that

$$(1+\lambda_{t+1})\gamma_t = \eta_t$$
 and $(1+\lambda_{t+1})\hat{\beta}_t = (1+\lambda_{t+1})\frac{\gamma_t\beta_t}{\gamma_{t-1}} = \lambda_t.$

7.2 Convergence for convex and smooth functions

First we provide a convergence theorem for any sequence of step sizes. Later we develop special cases of this theorem through different choices of the step sizes.

Theorem 7.4. Let Assumptions (Sum of L_{\max} -Smooth) and (Sum of Convex) hold. Consider $(x^t)_{t\in\mathbb{N}}$ the iterates generated by the (Momentum) algorithm with stepsize and momentum parameters taken according to

$$\gamma_t = \frac{2\eta}{t+3}, \quad \beta_t = \frac{t}{t+2}, \quad \text{with} \quad \eta \le \frac{1}{4L_{\max}}.$$

Then the iterates converge according to

$$\mathbb{E}\left[f(x^{t}) - \inf f\right] \le \frac{\|x_{0} - x^{*}\|^{2}}{\eta \left(t + 1\right)} + 2\eta \sigma_{f}^{*}.$$
(62)

Proof. For the proof, we rely on the iterate-moving-average (IMA) viewpoint of momentum given in Lemma 7.3. It is easy to verify that the parameters

$$\eta_t = \eta, \quad \lambda_t = \frac{t}{2} \quad \text{and} \quad z^t = x^{t+1} + \lambda_t (x^{t+1} - x^t)$$

verify the conditions of Lemma 7.3. Let us then consider the iterates (x^t, z^t) of (IMA), wand we start by studing the variations of $||z^t - x^*||^2$. Expanding squares we have for $t \in \mathbb{N}$ that

$$\begin{aligned} \|z^{t+1} - x^*\|^2 &= \|z^t - x^* - \eta \nabla f_{i_t}(x^t)\|^2 \\ \stackrel{(61)}{=} \|z^t - x^*\|^2 - 2\eta \langle \nabla f_{i_t}(x^t), z^t - x^* \rangle + \eta^2 \|\nabla f_{i_t}(x^t)\|^2 \\ \stackrel{(61)}{=} \|z^t - x^*\|^2 - 2\eta \langle \nabla f_{i_t}(x^t), x^t - x^* \rangle - 2\eta \lambda_t \langle \nabla f_{i_t}(x^t), x^t - x^{t-1} \rangle + \eta^2 \|\nabla f_{i_t}(x^t)\|^2. \end{aligned}$$

In the last equality we made appear x^{t-1} which , for t = 0, can be taken equal to zero. Then taking conditional expectation, using the convexity of f (via Lemma 2.8) and a variance transfer lemma (Lemma 4.20), we have

$$\mathbb{E}\left[\|z^{t+1} - x^*\|^2 | x^t \right]$$

$$= \|z^t - x^*\|^2 - 2\eta \langle \nabla f(x^t), x^t - x^* \rangle - 2\eta \lambda_t \langle \nabla f(x^t), x^t - x^{t-1} \rangle + \eta^2 \mathbb{E}_t \left[\|\nabla f_{i_t}(x^t)\|^2 | x^t \right],$$

$$\leq \|z^t - x^*\|^2 + (4\eta^2 L_{\max} - 2\eta) \left(f(x^t) - \inf f \right) - 2\eta \lambda_t \left(f(x^t) - f(x^{t-1}) \right) + 2\eta^2 \sigma_f^*$$

$$= \|z^t - x^*\|^2 - 2\eta \left(1 + \lambda_t - 2\eta L_{\max} \right) \left(f(x^t) - \inf f \right) + 2\eta \lambda_t \left(f(x^{t-1}) - \inf f \right) + 2\eta^2 \sigma_f^*$$

$$\leq \|z^t - x^*\|^2 - 2\eta \lambda_{t+1} \left(f(x^t) - \inf f \right) + 2\eta \lambda_t \left(f(x^{t-1}) - \inf f \right) + 2\eta^2 \sigma_f^*.$$

where we used the facts that $\eta \leq \frac{1}{4L_{\max}}$ and $\lambda_t + \frac{1}{2} = \lambda_{t+1}$ in the last inequality. Taking now expectation and summing over t = 0 to k, we have after telescoping and cancelling terms

$$\mathbb{E}\left[\|z^{k+1} - x^*\|^2\right] \le \|x_0 - x^*\|^2 - 2\eta\lambda_{k+1}\mathbb{E}\left[f(x^t) - \inf f\right] + 2(k+1)\sigma_f^*\eta^2,$$

where we used that $\lambda_0 = 0$. Reordering the terms, writing $2\lambda_t = t$ explicitly and getting rid of $\mathbb{E}\left[\|z^{t+1} - x^*\|^2\right]$, we finally obtain

$$\mathbb{E}\left[f(x^{t}) - \inf f\right] \le \frac{\|x_{0} - x^{*}\|^{2}}{\eta(t+1)} + 2\sigma_{f}^{*}\eta.$$

Corollary 7.5 ($\mathcal{O}(1/\epsilon^2)$ Complexity). Consider the setting of Theorem 7.4. We can guarantee that $\mathbb{E}\left[f(x^T) - \inf f\right] \leq \epsilon$ provided that we take

$$\eta = \frac{1}{\sqrt{T+1}} \frac{1}{4L_{\max}} \quad \text{and} \quad T \ge \frac{1}{\epsilon^2} \frac{1}{4L_{\max}^2} \left(\frac{1}{2} \|x_0 - x^*\|^2 + \sigma_f^*\right)^2 \tag{63}$$

Proof. By plugging $\eta = \frac{1}{\sqrt{T+1}} \frac{1}{4L_{\text{max}}}$ into (62), we obtain

$$\mathbb{E}\left[f(x_T) - \inf f\right] \le \frac{1}{2L_{\max}} \frac{1}{\sqrt{T+1}} \left(\frac{1}{2} \|x_0 - x^*\|^2 + \sigma_f^*\right).$$
(64)

The final complexity (63) follows by demanding that (64) be less than ϵ .

7.3 Bibliographic notes

This section is based on [39]. The deterministic momentum method was designed for strongly convex functions [34]. The authors in [12] showed that the deterministic momentum method converged globally and sublinearly for smooth and convex functions. Theorem 7.4 is from [39], which in turn is an extension of the results in [12]. For convergence proofs for momentum in the non-smooth setting see [7].

8 Theory : Nonsmooth functions

In this section we present the tools needed to handle nonsmooth functions. "Nonsmoothness" arise typically in two ways.

- 1. Continuous functions having points of nondifferentiability. For instance:
 - the L1 norm $||x||_1 = \sum_{i=1}^d |x_i|$. It is often used as a regularizer that promotes sparse minimizers.
 - the ReLU $\sigma(t) = 0$ if $t \le 0$, t if $t \ge 0$. It is often used as the activation function for neural networks, making the associated loss nondifferentiable.
- 2. Differentiable functions not being defined on the entire space. An other way to say it is that they take the value $+\infty$ outside of their domain. This can be seen as nonsmoothness, as the behaviour of the function at the boundary of the domain can be degenerate.
 - The most typical example is the indicator function of some constraint $C \subset \mathbb{R}^d$, and which is defined as $\delta_C(x) = 0$ if $x \in C$, $+\infty$ if $x \notin C$. Such function is useful because it allows to say that minimizing a function f over the constraint C is the same as minimizing the sum $f + \delta_C$.

8.1 Real-extended valued functions

Definition 8.1. Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$.

- 1. The **domain** of f is defined by dom $f \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d \mid f(x) < +\infty\}.$
- 2. We say that f is **proper** if dom $f \neq \emptyset$.

Definition 8.2. Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, and $\bar{x} \in \mathbb{R}^d$. We say that f is **lower semi-continuous** at \bar{x} if

$$f(\bar{x}) \le \liminf_{x \to \bar{x}} f(x).$$

We say that f is lower semi-continuous (**l.s.c.** for short) if f is lower semi-continuous at every $\bar{x} \in \mathbb{R}^d$.

Example 8.3 (Most functions are proper l.s.c.).

- If $f : \mathbb{R}^d \to \mathbb{R}$ is continuous, then it is proper and l.s.c.
- If $C \subset \mathbb{R}^d$ is closed and nonempty, then its indicator function δ_C is proper and l.s.c.
- A finite sum of proper l.s.c functions is proper l.s.c.

As hinted by the above example, it is safe to say that most functions used in practice are proper and l.s.c.. It is a minimal technical assumption which is nevertheless needed for what follows (see e.g. Lemmas 8.9 and 8.11).

8.2 Subdifferential of nonsmooth convex functions

We have seen in Lemma 2.8 that for *differentiable* convex functions, $\nabla f(x)$ verifies inequality (2). For non-differentiable (convex) functions f, we will use this fact as the basis to define a more general notion : *subgradients*.

Definition 8.4. Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, and $x \in \mathbb{R}^d$. We say that $\eta \in \mathbb{R}^d$ is a **subgradient** of f at $x \in \mathbb{R}^d$ if

for every
$$y \in \mathbb{R}^d$$
, $f(y) - f(x) - \langle \eta, y - x \rangle \ge 0.$ (65)

We denote by $\partial f(x)$ the set of all subgradients at x, that is :

$$\partial f(x) \stackrel{\text{def}}{=} \{ \eta \in \mathbb{R}^d \mid \text{ for all } y \in \mathbb{R}^d, \ f(y) - f(x) - \langle \eta, y - x \rangle \ge 0 \} \subset \mathbb{R}^d.$$
(66)

We also call $\partial f(x)$ the **subdifferential** of f. Finally, define dom $\partial f \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d \mid \partial f(x) \neq \emptyset\}$.

Subgradients are guaranteed to exist whenever f is convex and continuous.

Lemma 8.5. If $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a convex function and continuous at $x \in \mathbb{R}^d$, then $\partial f(x) \neq \emptyset$. This is always true if dom $f = \mathbb{R}^d$.

Proof. See [33, Proposition 3.25] and [2, Corollary 8.40].

If f is differentiable, then $\nabla f(x)$ is the unique subgradient at x, as we show next. This means that the subdifferential is a faithful generalization of the gradient.

Lemma 8.6. If $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a convex function that is is differentiable at $x \in \mathbb{R}^d$, then $\partial f(x) = \{\nabla f(x)\}.$

Proof. (Proof adapted from [33, Proposition 3.20]). From Lemma 2.8 we have that $\nabla f(x) \in \partial f(x)$. Suppose now that $\eta \in \partial f(x)$, and let us show that $\eta = \nabla f(x)$. For this, take any $v \in \mathbb{R}^d$ and t > 0, and Definition 8.4 to write

$$f(x+tv) - f(x) - \langle \eta, (x+tv) - x \rangle \ge 0 \quad \Leftrightarrow \quad \frac{f(x+tv) - f(x)}{t} \ge \langle \eta, v \rangle.$$

Taking the limit when $t \downarrow 0$, we obtain that

for all
$$v \in \mathbb{R}^d$$
, $\langle \nabla f(x), v \rangle \ge \langle \eta, v \rangle$.

By choosing $v = \eta - \nabla f(x)$, we obtain that $\|\nabla f(x) - \eta\|^2 \leq 0$ which in turn allows us to conclude that $\nabla f(x) = \eta$.

Remark 8.7. As hinted by the previous results and comments, this definition of subdifferential is tailored for nonsmooth *convex* functions. There exists other notions of subdifferential which are better suited for nonsmooth nonconvex functions. But we will not discuss it in this monograph, for the sake of simplicity. The reader interested in this topic can consult [6, 36].

Proposition 8.8 (Fermat's Theorem). Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, and $\bar{x} \in \mathbb{R}^d$. Then \bar{x} is a minimizer of f if and only if $0 \in \partial f(\bar{x})$.

Proof. From the Definition 8.4, we see that

$$\bar{x} \text{ is a minimizer of } f$$

$$\Leftrightarrow \quad \text{for all } y \in \mathbb{R}^d, f(y) - f(\bar{x}) \ge 0$$

$$\Leftrightarrow \quad \text{for all } y \in \mathbb{R}^d, f(y) - f(\bar{x}) - \langle 0, y - x \rangle \ge 0$$

$$\stackrel{(65)}{\Leftrightarrow} \quad 0 \in \partial f(\bar{x}).$$

Lemma 8.9 (Sum rule). Let $f : \mathbb{R}^d \to \mathbb{R}$ be convex and differentiable. Let $g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be proper l.s.c. convex. Then, for all $x \in \mathbb{R}^d$, $\partial(f+g)(x) = \{\nabla f(x)\} + \partial g(x)$.

Proof. See [33, Theorem 3.30].

Lemma 8.10 (Positive homogeneity). Let $f : \mathbb{R}^d \to \mathbb{R}$ be proper l.s.c. convex. Let $x \in \mathbb{R}^d$, and $\gamma \ge 0$. Then $\partial(\gamma f)(x) = \gamma \partial f(x)$.

Proof. It is an immediate consequence of Definition 8.4.

8.3 Nonsmooth strongly convex functions

In this context Lemma 2.13 remains true: Strongly convex functions do not need to be continuous to have a unique minimizer:

Lemma 8.11. If $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a proper l.s.c. μ -strongly convex function, then f admits a unique minimizer.

Proof. See [33, Corollary 2.20].

We also have an obvious analogue to Lemma 2.14:

Lemma 8.12. If $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a proper l.s.c and μ -strongly convex function, then for every $x, y \in \mathbb{R}^d$, and for every $\eta \in \partial f(x)$ we have that

$$f(y) - f(x) - \langle \eta, y - x \rangle \ge \frac{\mu}{2} ||y - x||^2.$$
 (67)

Proof. Define $g(x) := f(x) - \frac{\mu}{2} ||x||^2$. According to Lemma 2.12, g is convex. It is also clearly l.s.c. and proper, by definition. According to the sum rule in Lemma 8.9, we have $\partial f(x) = \partial g(x) + \mu x$. Therefore we can use the convexity of g with Definition 8.4 to write

$$f(y) - f(x) - \langle \eta, y - x \rangle \ge \frac{\mu}{2} \|y\|^2 - \frac{\mu}{2} \|x\|^2 - \langle \mu x, y - x \rangle = \frac{\mu}{2} \|y - x\|^2.$$

8.4 Proximal operator

In this section we study a key tool used in some algorithms for minimizing nonsmooth functions.

Definition 8.13. Let $g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a proper l.s.c convex function. We define the **proximal operator** of g as the function $\operatorname{prox}_g : \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$\operatorname{prox}_{g}(x) := \underset{x' \in \mathbb{R}^{d}}{\operatorname{argmin}} g(x') + \frac{1}{2} \|x' - x\|^{2}$$
(68)

The proximal operator is well defined because, since g(x') is convex the sum $g(x') + \frac{1}{2} ||x' - x||^2$ is strongly convex in x'. Thus there exists only one minimizer (recall Lemma 8.11).

Example 8.14 (Projection is a proximal operator). Let $C \subset \mathbb{R}^d$ be a nonempty closed convex set, and let δ_C be its indicator function. Then the proximal operator of δ_C is exactly the projection operator onto C:

$$\operatorname{prox}_{\delta_C}(x) = \operatorname{proj}_C(x) \stackrel{\text{def}}{=} \operatorname*{argmin}_{c \in C} \|c - x\|^2.$$

The proximal operator can be characterized with the subdifferential :

Lemma 8.15. Let $g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a proper l.s.c convex function, and let $x, p \in \mathbb{R}^d$. Then $p = \operatorname{prox}_q(x)$ if and only if

$$x - p \in \partial g(p). \tag{69}$$

Proof. From Definition 8.13 we know that $p = \text{prox}_f(x)$ if and only if p is the minimizer of $\phi(u) := g(u) + \frac{1}{2} ||u - x||^2$. From our hypotheses on g, it is clear that ϕ is proper l.s.c convex. So we can use Proposition 8.8 to say that it is equivalent to $0 \in \partial \phi(p)$. Moreover, we can use the sum rule from Lemma 8.9 to write that $\partial \phi(p) = \partial g(p) + \{p - x\}$. So we have proved that $p = \text{prox}_g(x)$ if and only if $0 \in \partial g(p) + \{p - x\}$, which is what we wanted to prove, after rearranging the terms. \Box

We show that, like the projection, the proximal operator is 1-Lipschitz (we also say that it is *non-expansive*). This property will be very interesting for some proofs since it will allow us to "get rid" of the proximal terms.

Lemma 8.16 (Non-expansiveness). Let $g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a proper l.s.c convex function. Then $\operatorname{prox}_q : \mathbb{R}^d \to \mathbb{R}^d$ is 1-Lipschitz :

for all
$$x, y \in \mathbb{R}^d$$
, $\|\operatorname{prox}_g(y) - \operatorname{prox}_g(x)\| \le \|y - x\|.$ (70)

Proof. Let $p_y \stackrel{\text{def}}{=} \operatorname{prox}_g(y)$ and $p_x \stackrel{\text{def}}{=} \operatorname{prox}_g(x)$. From $p_x = \operatorname{prox}_g(x)$ we have $x - p_x \in \partial g(p_x)$ (see Lemma 8.15), so from the definition of the subdifferential (Definition 8.4), we obtain

$$g(p_y) - g(p_x) - \langle x - p_x, p_y - p_x \rangle \ge 0.$$

Similarly, from $p_y = \operatorname{prox}_q(y)$ we also obtain

$$g(p_x) - g(p_y) - \langle y - p_y, p_x - p_y \rangle \ge 0.$$

Adding together the above two inequalities gives

$$\langle y - x - p_y + p_x, p_x - p_y \rangle \le 0.$$

Expanding the left argument of the inner product, and using the Cauchy-Schwartz inequality gives

$$||p_x - p_y||^2 \le \langle x - y, p_x - p_y \rangle \le ||x - y|| ||p_x - p_y||$$

Dividing through by $||p_x - p_y||$ (assuming this is non-zero otherwise (70) holds trivially) we have (70).

We end this section with an important property of the proximal operator : it can help to characterize the minimizers of composite functions as fixed points.

Lemma 8.17. Let $f : \mathbb{R}^d \to \mathbb{R}$ be convex differentiable, let $g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be proper l.s.c. convex. If $x^* \in \operatorname{argmin}(f+g)$, then

for all
$$\gamma > 0$$
, $\operatorname{prox}_{\gamma a}(x^* - \gamma \nabla f(x^*)) = x^*$.

Proof. Since $x^* \in \operatorname{argmin}(f+g)$ we have that $0 \in \partial(f+g)(x^*) = \nabla f(x^*) + \partial g(x^*)$ (Proposition 8.8 and Lemma 8.9). By multiplying both sides by γ then by adding x^* to both sides gives

$$(x^* - \gamma \nabla f(x^*)) - x^* \in \partial(\gamma g)(x^*)$$

According to Lemma 8.15, this means that $\operatorname{prox}_{\gamma g}(x^* - \gamma \nabla f(x^*)) = x^*$.

8.5 Controlling the variance

Definition 8.18. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a differentiable function. We define the (Bregman) **divergence** of f between y and x as

$$D_f(y;x) \stackrel{\text{def}}{=} f(y) - f(x) - \langle \nabla f(x), y - x \rangle.$$
(71)

Note that the divergence $D_f(y; x)$ is always nonegative when f is convex due to Lemma 2.8. Moreover, the divergence is also upper bounded by suboptimality.

Lemma 8.19. Let $f : \mathbb{R}^d \to \mathbb{R}$ be convex differentiable, and $g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be proper l.s.c. convex, and F = g + f. Then, for all $x^* \in \operatorname{argmin} F$, for all $x \in \mathbb{R}^d$,

$$0 \le D_f(x; x^*) \le F(x) - \inf F.$$

Proof. Since $x^* \in \operatorname{argmin} F$, we can use the Fermat Theorem (Proposition 8.8) and the sum rule (Lemma 8.9) to obtain the existence of some $\eta^* \in \partial g(x^*)$ such that $\nabla f(x^*) + \eta^* = 0$. Use now the definition of the Bregman divergence, and the convexity of g (via Lemma 2.8) to write

$$D_f(x;x^*) = f(x) - f(x^*) - \langle \nabla f(x^*), x - x^* \rangle = f(x) - f(x^*) + \langle \eta^*, x - x^* \rangle$$

$$\leq f(x) - f(x^*) + g(x) - g(x^*)$$

$$= F(x) - F(x^*).$$

Next we provide a variance transfer lemma, generalizing Lemma 4.20, which will prove to be useful when dealing with nonsmooth sum of functions in Section 11.

Lemma 8.20 (Variance transfer - General convex case). Let f verify Assumptions (Sum of L_{\max} -Smooth) and (Sum of Convex). For every $x, y \in \mathbb{R}^d$, we have

$$\mathbb{V}\left[\nabla f_i(x)\right] \le 4L_{\max}D_f(x;y) + 2\mathbb{V}\left[\nabla f_i(y)\right].$$

Proof. Simply use successively Lemma 4.6, the inequality $||a + b||^2 \leq 2||a||^2 + 2||b||^2$, and the expected smoothness (via Lemma 4.7) to write:

$$\begin{aligned} \mathbb{V}\left[\nabla f_{i}(x)\right] &\leq \mathbb{E}\left[\|\nabla f_{i}(x) - \nabla f(y)\|^{2}\right] \\ &\leq 2\mathbb{E}\left[\|\nabla f_{i}(x) - \nabla f_{i}(y)\|^{2}\right] + 2\mathbb{E}\left[\|\nabla f_{i}(y) - \nabla f(y)\|^{2}\right] \\ &= 2\mathbb{E}\left[\|\nabla f_{i}(x) - \nabla f_{i}(y)\|^{2}\right] + 2\mathbb{V}\left[\nabla f_{i}(y)\right] \\ &\leq 4L_{\max}D_{f}(x;y) + 2\mathbb{V}\left[\nabla f_{i}(y)\right]. \end{aligned}$$

Definition 8.21. Let $f : \mathbb{R}^d \to \mathbb{R}$ verify Assumption (Sum of L_{\max} -Smooth). Let $g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be proper l.s.c convex. Let F = g + f be such that argmin $F \neq \emptyset$. We define the composite gradient noise as follows

$$\sigma_F^* \stackrel{\text{def}}{=} \inf_{x^* \in \operatorname{argmin} F} \mathbb{V} \left[\nabla f_i(x^*) \right].$$
(72)

Note the difference between σ_f^* introduced in Definition 4.16 and σ_F^* introduced here is that the variance of gradients taken at the minimizers of the composite sum F, as opposed to f.

Lemma 8.22. Let $f : \mathbb{R}^d \to \mathbb{R}$ verify Assumptions (Sum of L_{\max} -Smooth) and (Sum of Convex). Let $g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be proper l.s.c convex. Let F = g + f be such that argmin $F \neq \emptyset$.

- 1. $\sigma_F^* \ge 0$.
- 2. $\sigma_F^* = \mathbb{V}[\nabla f_i(x^*)]$ for every $x^* \in \operatorname{argmin} F$.
- 3. If $\sigma_F^* = 0$ then there exists $x^* \in \operatorname{argmin} F$ such that $x^* \in \operatorname{argmin} (g+f_i)$ for all $i = 1, \ldots, n$. The converse implication is also true if g is differentiable at x^* .
- 4. $\sigma_F^* \leq 4L_{\max} \left(f(x^*) \inf f \right) + 2\sigma_f^*$, for every $x^* \in \operatorname{argmin} F$.

Proof. Item 1 is trivial. For item 2, consider two minimizers $x^*, x' \in \operatorname{argmin} F$, and use the expected smoothness of f (via Lemma 4.7) together with Lemma 8.19 to write

$$\frac{1}{2L_{\max}} \mathbb{E}\left[\|\nabla f_i(x^*) - \nabla f_i(x')\|^2 \right] \le D_f(x^*; x') \le F(x^*) - \inf F = 0.$$

In other words, we have $\nabla f_i(x^*) = \nabla f_i(x')$ for all i = 1, ..., n, which means that indeed $\mathbb{V}[\nabla f_i(x^*)] = \mathbb{V}[\nabla f_i(x')]$. Now we turn to item 3, and start by assuming that $\sigma_F^* = 0$. Let $x^* \in \operatorname{argmin} F$, and we know from the previous item that $\mathbb{V}[\nabla f_i(x^*)] = 0$. This is equivalent to say that, for every i,

 $\nabla f_i(x^*) = \nabla f(x^*)$. But x^* being a minimizer implies that $-\nabla f(x^*) \in \partial g(x^*)$ (use Proposition 8.8 and Lemma 8.9). So we have that $-\nabla f_i(x^*) \in \partial g(x^*)$, from which we conclude by the same arguments that $x^* \in \operatorname{argmin}(g + f_i)$. Now let us prove the converse implication, by assuming further that g is differentiable at x^* . From the assumption $x^* \in \operatorname{argmin}(g + f_i)$, we deduce that $-\nabla f_i(x^*) \in \partial g(x^*) = \nabla g(x^*)$ (see Lemma 8.6). Taking the expectation on this inequality also gives us that $-\nabla f(x^*) = \nabla g(x^*)$. In other words, $\nabla f_i(x^*) = \nabla f(x^*)$ for every i. We can then conclude that $\mathbb{V}[\nabla f_i(x^*)] = 0$. We finally turn to item 4, which is a direct consequence of Lemma 8.20 (with $x = x^* \in \operatorname{argmin} F$ and $y = x_f^* \in \operatorname{argmin} f$):

$$\sigma_F^* = \mathbb{V}\left[\nabla f_i(x^*)\right]$$

$$\leq 4L_{\max}D_f(x^*;x_f^*) + 2\mathbb{V}\left[\nabla f_i(x_f^*)\right]$$

$$= 4L_{\max}\left(f(x^*) - \inf f\right) + 2\sigma_f^*.$$

9 Stochastic Subgradient Descent

Problem 9.1 (Stochastic Function). We want to minimize a function $f : \mathbb{R}^d \to \mathbb{R}$ which writes as

$$f(x) \stackrel{\text{def}}{=} \mathbb{E}_{\mathcal{D}} \left[f_{\xi}(x) \right],$$

where \mathcal{D} is some distribution over \mathbb{R}^q , $\xi \in \mathbb{R}^q$ is sampled from \mathcal{D} , and $f_{\xi} : \mathbb{R}^d \to \mathbb{R}$. We require that the problem is well-posed, in the sense that argmin $f \neq \emptyset$.

In this section we assume that the functions f_{ξ} are convex and have bounded subgradients.

Assumption 9.2 (Stochastic convex and G-bounded subgradients). We consider the problem (Stochastic Function) and assume that

- (convexity) for every $\xi \in \mathbb{R}^q$, the function f_{ξ} is convex ;
- (subgradient selection) there exists a function $g : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^d$ such that, for every $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^q, g(x,\xi) \in \partial f_{\mathcal{E}}(x)$ and $\mathbb{E}_{\mathcal{D}}[g(x,\xi)]$ is well-defined;
- (bounded subgradients) there exists G > 0 such that, for all $x \in \mathbb{R}^d$, $\mathbb{E}_{\mathcal{D}}\left[\|g(x,\xi)\|^2\right] \leq G^2$.

We now define the Stochastic Subgradient Descent algorithm, which is an extension of (SGD). Instead of considering the gradient of a function f_i , we consider here some subgradient of f_{ξ} .

Algorithm 9.3 (SSD). Consider Problem (Stochastic Function) and let Assumption (Stochastic convex and *G*-bounded subgradients) hold. Let $x^0 \in \mathbb{R}^d$, and let $\gamma_t > 0$ be a sequence of stepsizes.

The Stochastic Subgradient Descent (SSD) algorithm is given by the iterates $(x^t)_{t\in\mathbb{N}}$ where

$$\xi_t \in \mathbb{R}^q$$
 Sampled i.i.d. $\xi_t \sim \mathcal{D}$ (73)

$$x^{t+1} = x^t - \gamma_t g(x^t, \xi_t), \qquad \text{with } g(x^t, \xi_t) \in \partial f_{\xi_t}(x^t).$$
(74)

In (SGD), the sampled subgradient $g(x^t, \xi_t)$ is an unbiased estimator of a subgradient of f at x^t .

Lemma 9.4. If Assumption (Stochastic convex and *G*-bounded subgradients) holds then for all $x \in \mathbb{R}^d$, we have $\mathbb{E}_{\mathcal{D}}[g(x,\xi)] \in \partial f(x)$.

Proof. With Assumption (Stochastic convex and *G*-bounded subgradients) we can use the fact that $g(x,\xi) \in \partial f_{\xi}(x)$ to write

$$f_{\xi}(y) - f_{\xi}(x) - \langle g(x,\xi), y - x \rangle \ge 0.$$

Taking expectation with respect to $\xi \sim \mathcal{D}$, and using the fact that $\mathbb{E}_{\mathcal{D}}[g(x,\xi)]$ is well-defined, we obtain

$$f(y) - f(x) - \langle \mathbb{E}_{\mathcal{D}} \left[g(x,\xi) \right], y - x \rangle \ge 0,$$

which proves that $\mathbb{E}_{\mathcal{D}}[g(x,\xi)] \in \partial f(x)$.

9.1 Convergence for convex functions and bounded gradients

Theorem 9.5. Let Assumption (Stochastic convex and *G*-bounded subgradients) hold, and consider $(x^t)_{t\in\mathbb{N}}$ a sequence generated by the (SSD) algorithm, with a sequence of stepsizes $\gamma_t > 0$. Then for $\bar{x}^T \stackrel{\text{def}}{=} \frac{1}{\sum_{k=0}^{T-1} \gamma_k} \sum_{t=0}^{T-1} \gamma_t x_t$ we have

$$\mathbb{E}_{\mathcal{D}}\left[f(\bar{x}^{T}) - \inf f\right] \le \frac{\|x^{0} - x^{*}\|^{2}}{2\sum_{t=0}^{T-1} \gamma_{k}} + \frac{\sum_{t=0}^{T-1} \gamma_{t}^{2} G^{2}}{2\sum_{t=0}^{T-1} \gamma_{k}}.$$
(75)

Proof. Expanding the squares we have that

$$\begin{aligned} \|x^{t+1} - x^*\|^2 &\stackrel{(SSD)}{=} \|x^t - x^* - \gamma_t g(x^t, \xi_t)\|^2 \\ &= \|x^t - x^*\|^2 - 2\gamma_t \left\langle g(x^t, \xi_t), x^t - x^* \right\rangle + \gamma_t^2 \|g(x^t, \xi_t)\|^2. \end{aligned}$$

We will use the fact that our subgradients are bounded from Assumption (Stochastic convex and G-bounded subgradients), and that $\mathbb{E}_{\mathcal{D}}\left[g(x^t,\xi_t) \mid x^t\right] \in \partial f(x^t)$ (see Lemma 9.4). Taking expectation conditioned on x^t we have that

$$\mathbb{E}_{\mathcal{D}}\left[\|x^{t+1} - x^*\|^2 \,|\, x^t\right] = \|x^t - x^*\|^2 - 2\gamma_t \left\langle \mathbb{E}_{\mathcal{D}}\left[g(x^t, \xi_t) \,|\, x^t\right], x^t - x^*\right\rangle + \gamma_t^2 \mathbb{E}_{\mathcal{D}}\left[\|g(x^t, \xi_t)\|^2 \,|\, x^t\right] \\ \leq \|x^t - x^*\|^2 - 2\gamma_t \left\langle \mathbb{E}_{\mathcal{D}}\left[g(x^t, \xi_t) \,|\, x^t\right], x^t - x^*\right\rangle + \gamma_t^2 G^2 \\ \stackrel{(65)}{\leq} \|x^t - x^*\|^2 - 2\gamma_t (f(x^t) - \inf f) + \gamma_t^2 G^2.$$
(76)

Re-arranging, taking expectation and summing up from t = 0, ..., T - 1 gives

$$2\sum_{t=0}^{T-1} \gamma_t \mathbb{E}_{\mathcal{D}} \left[f(x^t) - \inf f \right] \leq \sum_{t=0}^{T-1} \left(\mathbb{E}_{\mathcal{D}} \left[\|x^t - x^*\|^2 \right] - \mathbb{E}_{\mathcal{D}} \left[\|x^{t+1} - x^*\|^2 \right] \right) + \sum_{t=0}^{T-1} \gamma_t^2 G^2$$
$$= \mathbb{E}_{\mathcal{D}} \left[\|x^0 - x^*\|^2 \right] - \mathbb{E}_{\mathcal{D}} \left[\|x^T - x^*\|^2 \right] + \sum_{t=0}^{T-1} \gamma_t^2 G^2$$
$$\leq \|x^0 - x^*\|^2 + \sum_{t=0}^{T-1} \gamma_t^2 G^2. \tag{77}$$

Let $\bar{x}^T = \frac{1}{\sum_{k=0}^{T-1} \gamma_k} \sum_{t=0}^{T-1} \gamma_t x_t$. Dividing through by $2 \sum_{k=0}^{T-1} \gamma_k$ and using Jensen's inequality we have

$$\mathbb{E}_{\mathcal{D}}\left[f(\bar{x}^{T})\right] - \inf f \stackrel{\text{Jensen}}{\leq} \frac{1}{\sum_{k=0}^{T-1} \gamma_{k}} \sum_{t=0}^{T-1} \gamma_{t} \mathbb{E}_{\mathcal{D}}\left[f(x^{t})\right] - \inf f) \\
\leq \frac{\|x^{0} - x^{*}\|^{2}}{2\sum_{k=0}^{T-1} \gamma_{k}} + \frac{\sum_{t=0}^{T-1} \gamma_{t}^{2} G^{2}}{2\sum_{k=0}^{T-1} \gamma_{k}}.$$
(78)

In the next Theorem we specialize our previous estimate by considering a suitably decreasing sequence of stepsize from which we obtain a convergence rate $\mathcal{O}\left(\frac{\ln(T)}{\sqrt{T}}\right)$. In Corollary 9.7 we will consider a suitable constant stepsize leading to a finite-horizon rate of $\mathcal{O}\left(\frac{1}{\sqrt{T}}\right)$, which will traduce in a $\mathcal{O}\left(\frac{1}{\epsilon^2}\right)$ complexity.

Theorem 9.6. Let Assumption (Stochastic convex and *G*-bounded subgradients) hold, and consider $(x^t)_{t\in\mathbb{N}}$ a sequence generated by the (SSD) algorithm, with a sequence of stepsizes $\gamma_t \stackrel{\text{def}}{=} \frac{\gamma}{\sqrt{t+1}}$ for some $\gamma > 0$. We have for $\bar{x}^T \stackrel{\text{def}}{=} \frac{1}{\sum_{k=0}^{T-1} \gamma_k} \sum_{t=0}^{T-1} \gamma_t x_t$ that

$$\mathbb{E}_{\mathcal{D}}\left[f(\bar{x}_{T})\right] - \inf f \leq \frac{\|x^{0} - x^{*}\|^{2}}{4\gamma} \frac{1}{\sqrt{T} - 1} + \frac{\gamma G^{2}}{4} \frac{\log(T)}{\sqrt{T} - 1}$$

Proof. Start considering $\gamma_t = \frac{\gamma}{\sqrt{t+1}}$, and use integral bounds to write

$$\sum_{t=0}^{T-1} \frac{1}{\sqrt{t+1}} \ge \int_{s=1}^{T-1} (s+1)^{-1/2} ds = 2(\sqrt{T}-1).$$
$$\sum_{t=0}^{T-1} \frac{1}{t+1} \le \int_{s=0}^{T-1} (s+1)^{-1} ds = \log(T).$$

Plugging this into (75) gives the result

$$\mathbb{E}_{\mathcal{D}}\left[f(\bar{x}_{T})\right] - \inf f \leq \frac{1}{4\gamma} \frac{\|x^{0} - x^{*}\|^{2}}{\sqrt{T} - 1} + \frac{\gamma}{4} \frac{\log(T)G^{2}}{\sqrt{T} - 1}.$$
(79)

Now consider the constant stepsize $\gamma_t = \frac{\gamma}{\sqrt{T}}$. Since $\sum_{k=0}^{T-1} \gamma_k = \gamma \sqrt{T}$ and $\sum_{t=0}^{T-1} \gamma_t^2 = \gamma^2$, we deduce from (75) that

$$\mathbb{E}_{\mathcal{D}}\left[f(\bar{x}_T)\right] - \inf f \leq \frac{\|x^0 - x^*\|^2}{4\gamma\sqrt{T}} + \frac{\gamma G^2}{4\sqrt{T}}.$$
(80)

Corollary 9.7. Consider the setting of Theorem 9.5. We can guarantee that $\mathbb{E}_{\mathcal{D}}\left[f(\bar{x}^T) - \inf f\right] \leq 1$ ε provided that

$$T \ge \frac{\|x^0 - x^*\|G}{\epsilon^2} \quad \text{and} \quad \gamma_t \equiv \frac{\|x^0 - x^*\|^2 G}{\sqrt{T}}.$$

Proof. Consider the result of Theorem 9.5 with a constant stepsize $\gamma_t \equiv \gamma > 0$:

$$\mathbb{E}_{\mathcal{D}}\left[f(\bar{x}^{T}) - \inf f\right] \le \frac{\|x^{0} - x^{*}\|^{2}}{2\gamma T} + \frac{\gamma^{2}TG^{2}}{2\gamma T} = \frac{\|x^{0} - x^{*}\|^{2}}{2\gamma T} + \frac{\gamma G^{2}}{2}$$

It is a simple exercise to see that the above right-hand term is minimal when $\gamma = \frac{\|x^0 - x^*\|^2 G}{\sqrt{T}}$. In this case, we obtain

$$\mathbb{E}_{\mathcal{D}}\left[f(\bar{x}^{T}) - \inf f\right] \leq \frac{\|x^{0} - x^{*}\|^{2}G}{\sqrt{T}}.$$

g
$$\frac{\|x^{0} - x^{*}\|^{2}G}{\sqrt{T}} \leq \varepsilon \Leftrightarrow T \geq \frac{\|x^{0} - x^{*}\|G}{\epsilon^{2}}.$$

The result follows by writing

9.2 Convergence for strongly convex functions and bounded gradients

Theorem 9.8. Let Assumption (Stochastic convex and G-bounded subgradients) hold and assume further that f is μ -strongly convex. Consider $(x^t)_{t\in\mathbb{N}}$ a sequence generated by the (SSD) algorithm, with a constant stepsize $\gamma_t \equiv \gamma \in]0, \frac{1}{\mu}[$. The iterates satisfy

$$\mathbb{E}_{\mathcal{D}}\left[\|x^{t} - x^{*}\|^{2}\right] \leq (1 - \gamma\mu)^{t} \|x^{0} - x^{*}\|^{2} + \frac{\gamma G^{2}}{\mu}.$$

Proof. Expanding the squares we have that

$$||x^{t+1} - x^*||^2 \stackrel{(SSD)}{=} ||x^t - x^* - \gamma g(x^t, \xi_t)||^2$$

= $||x^t - x^*||^2 - 2\gamma \langle g(x^t, \xi_t), x^t - x^* \rangle + \gamma^2 ||g(x^t, \xi_t)||^2.$

We will now use the fact that our subgradients are bounded from Assumption (Stochastic convex and G-bounded subgradients). Moreover we use the fact that $\mathbb{E}_{\mathcal{D}}\left[g(x^t,\xi_t) \mid x^t\right] \in \partial f(x^t)$ (see Lemma 9.4), together with the fact that f is strongly convex. Taking expectation conditioned on x^t , we have that

$$\mathbb{E}_{\mathcal{D}}\left[\|x^{t+1} - x^*\|^2 \,|\, x^t\right] = \|x^t - x^*\|^2 - 2\gamma \left\langle \mathbb{E}_{\mathcal{D}}\left[g(x^t, \xi_t) \,|\, x^t\right], x^t - x^*\right\rangle + \gamma^2 \mathbb{E}_{\mathcal{D}}\left[\|g(x^t, \xi_t)\|^2 \,|\, x^t\right] \\ \leq \|x^t - x^*\|^2 - 2\gamma \left\langle \mathbb{E}_{\mathcal{D}}\left[g(x^t, \xi_t) \,|\, x^t\right], x^t - x^*\right\rangle + \gamma^2 G^2 \\ \stackrel{(67)}{\leq} \|x^t - x^*\|^2 - 2\gamma (f(x^t) - \inf f) - \gamma u\|x^t - x^*\|^2 + \gamma^2 G^2 \end{cases}$$
(81)

$$\leq \|x^{\iota} - x^{*}\|^{2} - 2\gamma(f(x^{\iota}) - \inf f) - \gamma\mu\|x^{\iota} - x^{*}\|^{2} + \gamma^{2}G^{2}$$

$$\leq (1 - \gamma\mu)\|x^{t} - x^{*}\|^{2} + \gamma^{2}G^{2}$$
(81)
(82)

$$(1 - \gamma \mu) \|x^{t} - x^{*}\|^{2} + \gamma^{2} G^{2}$$
(82)

Taking expectation on the above, and using a recurrence argument, we can deduce that

$$\mathbb{E}_{\mathcal{D}}\left[\|x^{t} - x^{*}\|^{2}\right] \leq (1 - \gamma\mu)^{t} \|x^{0} - x^{*}\|^{2} + \sum_{k=0}^{t-1} (1 - \gamma\mu)^{t} \gamma^{2} G^{2}$$

Since

$$\sum_{k=0}^{t-1} (1 - \gamma \mu)^t \gamma^2 G^2 = \gamma^2 G^2 \frac{1 - (1 - \gamma \mu)^t}{\gamma \mu} \le \frac{\gamma^2 G^2}{\gamma \mu} = \frac{\gamma G^2}{\mu},$$
(83)

we conclude that

$$\mathbb{E}_{\mathcal{D}}\left[\|x^{t} - x^{*}\|^{2}\right] \leq (1 - \gamma\mu)^{t} \|x^{0} - x^{*}\|^{2} + \frac{\gamma G^{2}}{\mu}.$$

9.3 Better convergence rates for convex functions with bounded solution

In the previous section, we saw that (SSD) has a $\mathcal{O}\left(\frac{\ln(T)}{\sqrt{T}}\right)$ convergence rate, but enjoys a $\mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$ complexity rate. The latter suggests that it is possible to get rid of the logarithmic term and achieve a $\mathcal{O}\left(\frac{1}{\sqrt{T}}\right)$ convergence rate. In this section, we see that this can be done, by making a localization assumption on the solution of the problem, and by making a slight modification to the (SSD) algorithm.

Assumption 9.9 (*B*-Bounded Solution). There exists B > 0 and a solution $x^* \in \operatorname{argmin} f$ such that $||x^*|| \leq B$.

We will exploit this assumption by modifying the (SSD) algorithm, adding a projection step onto the closed ball $\mathbb{B}(0, B)$ where we know that the solution belongs. In this case the projection onto the ball is given by

$$\operatorname{proj}_{\mathbb{B}(0,B)}(x) := \begin{cases} x & \text{if } \|x\| \le B, \\ x\frac{B}{\|x\|} & \text{if } \|x\| > B. \end{cases}$$

See Example 8.14 for the definition of the projection onto a closed convex set.

Algorithm 9.10 (PSSD). Consider Problem (Stochastic Function) and let Assumptions (Stochastic convex and *G*-bounded subgradients) and (*B*-Bounded Solution) hold. Let $x^0 \in \mathbb{R}^d$, and let $\gamma_t > 0$ be a sequence of stepsizes. The **Projected Stochastic Subgradient Descent (PSSD)** algorithm is given by the iterates $(x^t)_{t\in\mathbb{N}}$ where

$$\xi_t \in \mathbb{R}^q$$
 Sampled i.i.d. $\xi_t \sim \mathcal{D}$ (84)

$$x^{t+1} = \operatorname{proj}_{\mathbb{B}(0,B)}(x^t - \gamma_t g(x^t, \xi_t)), \qquad \text{with } g(x^t, \xi_t) \in \partial f_{\xi_t}(x^t).$$
(85)

We now prove the following theorem, which is a simplified version of Theorem 19 in [7].

Theorem 9.11. Let Assumptions (Stochastic convex and *G*-bounded subgradients) and (*B*-Bounded Solution) hold. Let $(x^t)_{t\in\mathbb{N}}$ be the iterates of (PSSD), with a decreasing sequence of stepsizes $\gamma_t \stackrel{\text{def}}{=} \frac{\gamma}{\sqrt{t+1}}$, with $\gamma > 0$. Then we have for $T \ge 2$ and $\bar{x}_T \stackrel{\text{def}}{=} \frac{1}{T} \sum_{t=0}^{T-1} x_t$ that

$$\mathbb{E}_{\mathcal{D}}\left[f(\bar{x}_T) - \inf f\right] \leq \frac{1}{\sqrt{T}} \left(\frac{3B^2}{\gamma} + \gamma G^2\right).$$

Proof. We start by using Assumption (*B*-Bounded Solution) to write $\operatorname{proj}_{\mathbb{B}(0,B)}(x^*) = x^*$. This together with the fact that the projection is nonexpansive (see Lemma 8.16 and Example 8.14) allows us to write, after expanding the squares

$$\begin{aligned} \|x^{t+1} - x^*\|^2 &\stackrel{(PSSD)}{=} \|\operatorname{proj}_{\mathbb{B}(0,B)}(x^t - \gamma_t g(x^t, \xi_t)) - \operatorname{proj}_{\mathbb{B}(0,B)}(x^*)\|^2 \\ &\leq \|x^t - x^* - \gamma_t g(x^t, \xi_t)\|^2 \\ &= \|x^t - x^*\|^2 - 2\gamma_t \left\langle g(x^t, \xi_t), x^t - x^* \right\rangle + \gamma_t^2 \|g(x^t, \xi_t)\|^2, \end{aligned}$$

We now want to take expectation conditioned on x^t . We will use the fact that our subgradients are bounded from Assumption (Stochastic convex and *G*-bounded subgradients), and that $\mathbb{E}_{\mathcal{D}}\left[g(x^t,\xi_t) \mid x^t\right] \in \partial f(x^t)$ (see Lemma 9.4).

$$\mathbb{E}_{\mathcal{D}}\left[\|x^{t+1} - x^*\|^2 \,|\, x^t\right] = \|x^t - x^*\|^2 - 2\gamma_t \left\langle \mathbb{E}_{\mathcal{D}}\left[g(x^t, \xi_t) \mid x^t\right], x^t - x^*\right\rangle + \gamma_t^2 \mathbb{E}_{\mathcal{D}}\left[\|g(x^t, \xi_t)\|^2 \mid x^t\right] \\ \leq \|x^t - x^*\|^2 - 2\gamma_t \left\langle \mathbb{E}_{\mathcal{D}}\left[g(x^t, \xi_t) \mid x^t\right], x^t - x^*\right\rangle + \gamma_t^2 G^2 \\ \stackrel{(65)}{\leq} \|x^t - x^*\|^2 - 2\gamma_t (f(x^t) - \inf f) + \gamma_t^2 G^2.$$

Taking expectation, dividing through by $\frac{1}{2\gamma_t}$ and re-arranging gives

$$\mathbb{E}_{\mathcal{D}}\left[f(x^{t}) - \inf f\right] \leq \frac{1}{2\gamma_{t}} \mathbb{E}_{\mathcal{D}}\left[\|x^{t} - x^{*}\|^{2}\right] - \frac{1}{2\gamma_{t}} \mathbb{E}_{\mathcal{D}}\left[\|x^{t+1} - x^{*}\|^{2}\right] + \frac{\gamma_{t}G^{2}}{2}.$$

Summing up from t = 0, ..., T - 1 and using telescopic cancellation gives

$$\sum_{t=0}^{T-1} \mathbb{E}_{\mathcal{D}} \left[f(x^t) - \inf f \right] \le \frac{1}{2\gamma_0} \|x^0 - x^*\|^2 + \frac{1}{2} \sum_{t=0}^{T-2} \left(\frac{1}{\gamma_{t+1}} - \frac{1}{\gamma_t} \right) \mathbb{E}_{\mathcal{D}} \left[\|x^{t+1} - x^*\|^2 \right] + \frac{G^2}{2} \sum_{t=0}^{T-1} \gamma_t.$$

In the above inequality, we are going to bound the term

$$\left(\frac{1}{\gamma_{t+1}} - \frac{1}{\gamma_t}\right) = \frac{\sqrt{t+1} - \sqrt{t}}{\gamma} \le \frac{1}{2\gamma\sqrt{t}},$$

by using the fact that the square root function is concave. We are also going to bound the term $\mathbb{E}_{\mathcal{D}}\left[\|x^{t+1} - x^*\|^2\right]$ by using the fact that x^* and the sequence x^t belong to $\mathbb{B}(0, B)$, due to the projection step in the algorithm:

$$||x^{t} - x^{*}||^{2} \le ||x^{t}||^{2} + 2\langle x^{t}, x^{*} \rangle + ||x^{*}||^{2} \le 4B^{2}.$$

So we can now write (we use $\sqrt{2}-1 \leq \frac{1}{2})$:

$$\begin{split} \sum_{t=0}^{T-1} \mathbb{E}_{\mathcal{D}} \left[f(x^t) - \inf f \right] &\leq \frac{4B^2}{2\gamma} + \frac{1}{2} \left(\frac{\sqrt{2}}{\gamma} - \frac{1}{\gamma} \right) 4B^2 + \frac{1}{2} \sum_{t=1}^{T-2} \frac{4B^2}{2\gamma\sqrt{t}} + \frac{\gamma G^2}{2} \sum_{t=0}^{T-1} \frac{1}{\sqrt{t+1}} \\ &\leq \frac{3B^2}{\gamma} + \frac{B^2}{\gamma} \sum_{t=1}^{T} \frac{1}{\sqrt{t}} + \frac{\gamma G^2}{2} \sum_{t=1}^{T} \frac{1}{\sqrt{t}} \\ &\leq \frac{3B^2}{\gamma} + \left(\frac{B^2}{\gamma} + \frac{\gamma G^2}{2} \right) 2 \left(\sqrt{T} - 1 \right) \end{split}$$

where in the last inequality we used that

$$\sum_{t=1}^{T} \frac{1}{\sqrt{t}} \le \int_{s=1}^{T} s^{-1/2} ds = 2(\sqrt{T} - 1)$$

Cancelling some negative terms, and using the fact that $\sqrt{T} \ge 1$, we end up with

$$\sum_{t=0}^{T-1} \mathbb{E}_{\mathcal{D}}\left[f(x^t) - \inf f\right] \le \left(\frac{2B^2}{\gamma} + \gamma G^2\right)\sqrt{T} + \frac{B^2}{\gamma} \le \left(\frac{3B^2}{\gamma} + \gamma G^2\right)\sqrt{T}.$$

Finally let $\bar{x}_T = \frac{1}{T} \sum_{t=0}^{T-1} x_t$, dividing through by 1/T, and using Jensen's inequality we have that

$$\mathbb{E}_{\mathcal{D}}\left[f(\bar{x}_{T}) - \inf f\right] \leq \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_{\mathcal{D}}\left[f(x^{t}) - \inf f\right]$$
$$\leq \left(\frac{3B^{2}}{\gamma} + \gamma G^{2}\right) \frac{1}{\sqrt{T}}.$$

9.4 Bibliographic notes

The earlier non-asymptotic proof for the non-smooth case first appeared in the online learning literature, see for example [42]. Outside of the online setting, convergence proofs for SGD in the non-smooth setting with Lipschitz functions was given in [41]. For the non-smooth strongly convex setting see [22] where the authors prove a simple 1/t convergence rate.

10 Proximal Gradient Descent

Problem 10.1 (Composite). We want to minimize a function $F : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ which is a composite sum given by

$$F(x) = f(x) + g(x),$$
 (86)

where $f : \mathbb{R}^d \to \mathbb{R}$ is differentiable, and $g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is proper l.s.c. We require that the problem is well-posed, in the sense that argmin $F \neq \emptyset$.

To exploit the structure of this composite sum, we will use the proximal gradient descent algorithm, which alternates gradient steps with respect to the differentiable term f, and proximal steps with respect to the nonsmooth term g.

Algorithm 10.2 (PGD). Let $x^0 \in \mathbb{R}^d$, and let $\gamma > 0$ be a stepsize. The **Proximal Gradient** Descent (PGD) algorithm defines a sequence $(x^t)_{t\in\mathbb{N}}$ which satisfies

$$x^{t+1} = \operatorname{prox}_{\gamma q}(x^t - \gamma \nabla f(x^t)).$$
(87)

10.1 Convergence for convex functions

Theorem 10.3. Consider the Problem (Composite), and suppose that g is convex, and that f is convex and L-smooth, for some L > 0. Let $(x^t)_{t \in \mathbb{N}}$ be the sequence of iterates generated by the algorithm (PGD), with a stepsize $\gamma \in]0, \frac{1}{L}]$. Then, for all $x^* \in \operatorname{argmin} F$, for all $t \in \mathbb{N}$ we have that

$$F(x^t) - \inf F \le \frac{\|x^0 - x^*\|^2}{2\gamma t}.$$

Proof. Let $x^* \in \operatorname{argmin} F$ be any minimizer of F. We start by studying two (decreasing and nonnegative) quantities of interest : $F(x^t) - \inf F$ and $||x^t - x^*||^2$.

First, we show that $F(x^{t+1}) - \inf F$ decreases. For this, using the definition of $\operatorname{prox}_{\gamma g}$ in (68) together with the update rule (87), we have that

$$x^{t+1} = \underset{x' \in \mathbb{R}^d}{\operatorname{argmin}} \ \frac{1}{2} \|x' - (x^t - \gamma \nabla f(x^t))\|^2 + \gamma g(x').$$

Consequently

$$g(x^{t+1}) + \frac{1}{2\gamma} \|x^{t+1} - (x^t - \gamma \nabla f(x^t))\|^2 \le g(x^t) + \frac{1}{2\gamma} \|x^t - (x^t - \gamma \nabla f(x^t))\|^2.$$

After expanding the squares and rearranging the terms, we see that the above inequality is equivalent to

$$g(x^{t+1}) - g(x^t) \le \frac{-1}{2\gamma} \|x^{t+1} - x^t\|^2 - \langle \nabla f(x^t), x^{t+1} - x^t \rangle.$$
(88)

Now, we can use the fact that f is L-smooth and (9) to write

$$f(x^{t+1}) - f(x^t) \le \langle \nabla f(x^t), x^{t+1} - x^t \rangle + \frac{L}{2} \|x^{t+1} - x^t\|^2.$$
(89)

Summing (88) and (89), and using the fact that $\gamma L \leq 1$, we obtain that

$$F(x^{t+1}) - F(x^t) \le \frac{-1}{2\gamma} \|x^{t+1} - x^t\|^2 + \frac{L}{2} \|x^{t+1} - x^t\|^2 \le 0.$$
(90)

Consequently $F(x^{t+1}) - F(x^t)$ is decreasing.

Now we show that $||x^{t+1} - x^*||^2$ is decreasing. For this we first expand the squares as follows

$$\frac{1}{2\gamma} \|x^{t+1} - x^*\|^2 - \frac{1}{2\gamma} \|x^t - x^*\|^2 = \frac{-1}{2\gamma} \|x^{t+1} - x^t\|^2 - \langle \frac{x^t - x^{t+1}}{\gamma}, x^{t+1} - x^* \rangle.$$
(91)

Since $x^{t+1} = \operatorname{prox}_{\gamma g}(x^t - \gamma \nabla f(x^t))$, we know from (69) that

$$\frac{x^t - x^{t+1}}{\gamma} \in \nabla f(x^t) + \partial g(x^{t+1}).$$

Using the above in (91) we have that there exists some $\eta^{t+1} \in \partial g(x^{t+1})$ such that

$$\begin{aligned} &\frac{1}{2\gamma} \|x^{t+1} - x^*\|^2 - \frac{1}{2\gamma} \|x^t - x^*\|^2 \\ &= \frac{-1}{2\gamma} \|x^{t+1} - x^t\|^2 - \langle \nabla f(x^t) + \eta^{t+1}, x^{t+1} - x^* \rangle \\ &= \frac{-1}{2\gamma} \|x^{t+1} - x^t\|^2 - \langle \eta^{t+1}, x^{t+1} - x^* \rangle - \langle \nabla f(x^t), x^{t+1} - x^t \rangle + \langle \nabla f(x^t), x^* - x^t \rangle. \end{aligned}$$

On the first inner product term we can use that $\eta^{t+1} \in \partial g(x^{t+1})$ and definition of subgradient (65) to write

$$-\langle \eta^{t+1}, x^{t+1} - x^* \rangle = \langle \eta^{t+1}, x^* - x^{t+1} \rangle \le g(x^*) - g(x^{t+1}).$$
(92)

On the second inner product term we can use the smoothness of L and (9) to write

$$-\langle \nabla f(x^{t}), x^{t+1} - x^{t} \rangle \leq \frac{L}{2} \|x^{t+1} - x^{t}\|^{2} + f(x^{t}) - f(x^{t+1}).$$
(93)

On the last term we can use the convexity of f and (2) to write

$$\langle \nabla f(x^t), x^* - x^t \rangle \le f(x^*) - f(x^t).$$
(94)

By combining (92), (93), (94), and using the fact that $\gamma L \leq 1$, we obtain

$$\frac{1}{2\gamma} \|x^{t+1} - x^*\|^2 - \frac{1}{2\gamma} \|x^t - x^*\|^2 \leq \frac{-1}{2\gamma} \|x^{t+1} - x^t\|^2 + \frac{L}{2} \|x^{t+1} - x^t\|^2
+ g(x^*) - g(x^{t+1}) + f(x^t) - f(x^{t+1}) + f(x^*) - f(x^t)
= \frac{-1}{2\gamma} \|x^{t+1} - x^t\|^2 + \frac{L}{2} \|x^{t+1} - x^t\|^2 - (F(x^{t+1}) - F(x^*))
\leq -(F(x^{t+1}) - \inf F).$$
(95)

Now that we have established that the iterate gap and functions values are decreasing, we want to show that the Lyapunov energy

$$E_t := \frac{1}{2\gamma} \|x^t - x^*\|^2 + t(F(x^t) - \inf F),$$

is decreasing. Indeed, re-arranging the terms and using (90) and (95) we have that

$$E_{t+1} - E_t = (t+1)(F(x^{t+1}) - \inf F) - t(F(x^t) - \inf F) + \frac{1}{2\gamma} \|x^{t+1} - x^*\|^2 - \frac{1}{2\gamma} \|x^t - x^*\|^2$$

$$= F(x^{t+1}) - \inf F + t(F(x^{t+1}) - F(x^t)) + \frac{1}{2\gamma} \|x^{t+1} - x^*\|^2 - \frac{1}{2\gamma} \|x^t - x^*\|^2$$

$$\stackrel{(90)}{\leq} F(x^{t+1}) - \inf F + \frac{1}{2\gamma} \|x^{t+1} - x^*\|^2 - \frac{1}{2\gamma} \|x^t - x^*\|^2 \stackrel{(95)}{\leq} 0.$$
(96)

We have shown that E_t is decreasing, therefore we can write that

$$t(F(x^t) - \inf F) \leq E_t \leq E_0 = \frac{1}{2\gamma} ||x^0 - x^*||^2,$$

and the conclusion follows after dividing by t.

54

Corollary 10.4 ($\mathcal{O}(1/t)$ Complexity). Consider the setting of Theorem 10.3, for a given $\epsilon > 0$ and $\gamma = L$ we have that

$$t \ge \frac{L}{\epsilon} \frac{\|x^0 - x^*\|^2}{2} \implies F(x^t) - \inf F \le \epsilon$$
(97)

10.2 Convergence for strongly convex functions

Theorem 10.5. Consider the Problem (Composite), and suppose that h is convex, and that f is μ -strongly convex and L-smooth, for some $L \ge \mu > 0$. Let $(x^t)_{t \in \mathbb{N}}$ be the sequence of iterates generated by the algorithm (PGD), with a stepsize $0 < \gamma \le \frac{1}{L}$. Then, for $x^* = \operatorname{argmin} F$ and $t \in \mathbb{N}$ we have that

$$\|x^{t+1} - x^*\|^2 \leq (1 - \gamma\mu) \|x^t - x^*\|^2.$$
(98)

As for (GD) (see Theorem 3.6) we provide two different proofs here.

Proof of Theorem 10.5 with first-order properties. Use the definition of (PGD) together with Lemma 8.17, and the nonexpansiveness of the proximal operator (Lemma 8.16), to write

$$\begin{aligned} \|x^{t+1} - x^*\|^2 &\leq \|\operatorname{prox}_{\gamma g}(x^t - \gamma \nabla f(x^t)) - \operatorname{prox}_{\gamma g}(x^* - \gamma \nabla f(x^*))\|^2 \\ &\leq \|(x^t - x^*) - \gamma (\nabla f(x^t) - \nabla f(x^*))\|^2 \\ &= \|x^t - x^*\|^2 + \gamma^2 \|\nabla f(x^t) - \nabla f(x^*)\|^2 - 2\gamma \langle \nabla f(x^t) - \nabla f(x^*), x^t - x^* \rangle. \end{aligned}$$

The cocoercivity of f (Lemma 2.29) gives us

$$\gamma^2 \|\nabla f(x^t) - \nabla f(x^*)\|^2 \le 2\gamma^2 L \left(f(x^t) - f(x^*) - \langle \nabla f(x^*), x^t - x^* \rangle \right),$$

while the strong convexity of f gives us (Lemma 2.14)

$$\begin{aligned} -2\gamma \langle \nabla f(x^t) - \nabla f(x^*), x^t - x^* \rangle &= 2\gamma \langle \nabla f(x^t), x^* - x^t \rangle + 2\gamma \langle \nabla f(x^*), x^t - x^* \rangle \\ &\leq 2\gamma \left(f(x^*) - f(x^t) - \frac{\mu}{2} \|x^t - x^*\|^2 \right) + 2\gamma \langle \nabla f(x^*), x^t - x^* \rangle \\ &= -\gamma \mu \|x^t - x^*\|^2 - 2\gamma \left(f(x^t) - f(x^*) - \langle \nabla f(x^*), x^t - x^* \rangle \right) \end{aligned}$$

Combining those three inequalities and rearranging the terms, we obtain

$$\|x^{t+1} - x^*\|^2 \leq (1 - \gamma\mu)\|x^t - x^*\|^2 + (2\gamma^2 L - 2\gamma)\left(f(x^t) - f(x^*) - \langle \nabla f(x^*), x^t - x^* \rangle\right).$$

We conclude after observing that $f(x^t) - f(x^*) - \langle \nabla f(x^*), x^t - x^* \rangle \ge 0$ (because f is convex, see Lemma 2.8), and that $2\gamma^2 L - 2\gamma \le 0$ (because of our assumption on the stepsize).

Proof of Theorem 10.5 with the Hessian. Let $T(x) := x - \gamma \nabla f(x)$ so that the iterates of (PGD) verify $x^{t+1} = \operatorname{prox}_{\gamma g}(T(x^t))$. From Lemma 8.17 we know that $\operatorname{prox}_{\gamma g}(T(x^*)) = x^*$, so we can write

$$||x^{t+1} - x^*|| = ||\operatorname{prox}_{\gamma h}(T(x^t)) - \operatorname{prox}_{\gamma g}(T(x^*))||.$$

Moreover, we know from Lemma 8.16 that $\operatorname{prox}_{\gamma q}$ is 1-Lipschitz, so

$$||x^{t+1} - x^*|| \le ||T(x^t) - T(x^*)||$$

Further, we already proved in the proof of Theorem 3.6 that T is $(1 - \gamma \mu)$ -Lipschitz (assuming further that f is twice differentiable). Consequently,

$$||x^{t+1} - x^*|| \le (1 - \gamma \mu) ||x^t - x^*||$$

To conclude the proof, take the squares in the above inequality, and use the fact that $(1 - \gamma \mu)^2 \leq (1 - \gamma \mu)$.

Corollary 10.6 (log(1/ ϵ) Complexity). Consider the setting of Theorem 10.5, for a given $\epsilon > 0$, we have that if $\gamma = 1/L$ then

$$k \ge \frac{L}{\mu} \log\left(\frac{1}{\epsilon}\right) \quad \Rightarrow \quad \|x^{t+1} - x^*\|^2 \le \epsilon \|x^0 - x^*\|^2.$$
(99)

The proof of this lemma follows by applying Lemma A.1 in the appendix.

10.3 Bibliographic notes

A proof of a $\mathcal{O}(\frac{1}{T})$ convergence rate for the (PGD) algorithm in the convex case can be found in [3, Theorem 3.1]. The linear convergence rate in the strongly convex case can be found in [37, Proposition 4].

11 Stochastic Proximal Gradient Descent

Problem 11.1 (Composite Sum of Functions). We want to minimize a function $F : \mathbb{R}^d \to \mathbb{R}$ which writes as a composite sum

$$F(x) \stackrel{\text{def}}{=} g(x) + f(x), \quad f(x) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i=1}^{n} f_i(x),$$

where each $f_i : \mathbb{R}^d \to \mathbb{R}$ is differentiable, and $g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is proper l.s.c. We require that the problem is well-posed, in the sense that argmin $F \neq \emptyset$, and each f_i is bounded from below.

Assumption 11.2 (Composite Sum of Convex). We consider the Problem (Composite Sum of Functions) and we suppose that h and each f_i are convex.

Assumption 11.3 (Composite Sum of L_{\max} -Smooth). We consider the Problem (Composite Sum of Functions) and suppose that each f_i is L_i -smooth. We note $L_{\max} = \max_{i=1,\dots,n} L_i$.

Algorithm 11.4 (SPGD). Consider the Problem (Composite Sum of Functions). Let $x^0 \in \mathbb{R}^d$, and let $\gamma_t > 0$ be a sequence of step sizes. The Stochastic Proximal Gradient Descent (SPGD) algorithm defines a sequence $(x^t)_{t\in\mathbb{N}}$ satisfying

$$i_t \in \{1, \dots n\}$$
Sampled with probability $\frac{1}{n}$,
$$x^{t+1} = \operatorname{prox}_{\gamma g} \left(x^t - \gamma_t \nabla f_{i_t}(x^t) \right).$$
(100)

11.1 Complexity for convex functions

Theorem 11.5. Let Assumptions (Composite Sum of L_{\max} -Smooth) and (Composite Sum of Convex) hold. Let $(x^t)_{t\in\mathbb{N}}$ be a sequence generated by the (SPGD) algorithm with a nonincreasing sequence of stepsizes verifying $0 < \gamma_0 < \frac{1}{4L_{\max}}$. Then, for all $x^* \in \operatorname{argmin} F$, for all $t \in \mathbb{N}$,

$$\mathbb{E}\left[F(\bar{x}^{t}) - \inf F\right] \le \frac{\|x^{0} - x^{*}\|^{2} + 2\gamma_{0}(F(x^{0}) - \inf F)}{2(1 - 4\gamma_{0}L_{\max})\sum_{s=0}^{t-1}\gamma_{s}} + \frac{2\sigma_{F}^{*}}{(1 - 4\gamma_{0}L_{\max})}\frac{\sum_{s=0}^{t-1}\gamma_{s}^{2}}{\sum_{s=0}^{t-1}\gamma_{s}}$$

where $\bar{x}^t \stackrel{\text{def}}{=} \frac{1}{\sum_{s=0}^{t-1} \gamma_s} \sum_{s=0}^{t-1} \gamma_s x^s$.

Proof. Let us start by looking at $||x^{t+1} - x^*||^2 - ||x^t - x^*||^2$. Since we just compare x^t to x^{t+1} , to lighten the notations we fix $j := i_t$. Expanding the squares, we have that

$$\frac{1}{2\gamma_t} \|x^{t+1} - x^*\|^2 - \frac{1}{2\gamma_t} \|x^t - x^*\|^2 = \frac{-1}{2\gamma_t} \|x^{t+1} - x^t\|^2 - \langle \frac{x^t - x^{t+1}}{\gamma_t}, x^{t+1} - x^* \rangle = \frac{-1}{2\gamma_t} \|x^{t+1} - x^t\|^2 - \langle \frac{x^t - x^{t+1}}{\gamma_t}, x^{t+1} - x^* \rangle = \frac{-1}{2\gamma_t} \|x^{t+1} - x^t\|^2 - \langle \frac{x^t - x^{t+1}}{\gamma_t}, x^{t+1} - x^* \rangle = \frac{-1}{2\gamma_t} \|x^{t+1} - x^t\|^2 - \langle \frac{x^t - x^{t+1}}{\gamma_t}, x^{t+1} - x^* \rangle = \frac{-1}{2\gamma_t} \|x^{t+1} - x^t\|^2 - \langle \frac{x^t - x^{t+1}}{\gamma_t}, x^{t+1} - x^* \rangle = \frac{-1}{2\gamma_t} \|x^{t+1} - x^t\|^2 - \langle \frac{x^t - x^{t+1}}{\gamma_t}, x^{t+1} - x^* \rangle = \frac{-1}{2\gamma_t} \|x^{t+1} - x^t\|^2 + \frac{-1}{2\gamma_t} \|x^{t+1} - x^*\|^2 + \frac{-1}{2\gamma_t} \|x^{t+1} - x^t\|^2 + \frac{-1}{2\gamma_t} \|x^{t+1} - x^*\|^2 + \frac{-1}{2\gamma_t} \|x^{t+1} - x^t\|^2 + \frac{-1}{2\gamma_t} \|x^{t+1} - x^*\|^2 +$$

Since $x^{t+1} = \operatorname{prox}_{\gamma_t g}(x^t - \gamma_t \nabla f_j(x^t))$, we know from (69) that $\frac{x^t - x^{t+1}}{\gamma_t} \in \nabla f_j(x^t) + \partial g(x^{t+1})$. So there exists some $\eta^{t+1} \in \partial g(x^{t+1})$ such that

$$\frac{1}{2\gamma_t} \|x^{t+1} - x^*\|^2 - \frac{1}{2\gamma_t} \|x^t - x^*\|^2 \tag{101}$$

$$= \frac{-1}{2\gamma_t} \|x^{t+1} - x^t\|^2 - \langle \nabla f_j(x^t) + \eta^{t+1}, x^{t+1} - x^* \rangle$$

$$= \frac{-1}{2\gamma_t} \|x^{t+1} - x^t\|^2 - \langle \nabla f_j(x^t) - \nabla f(x^t), x^{t+1} - x^* \rangle - \langle \nabla f(x^t) + \eta^{t+1}, x^{t+1} - x^* \rangle$$

We decompose the last term of (101) as

$$-\langle \nabla f(x^{t}) + \eta^{t+1}, x^{t+1} - x^{*} \rangle = -\langle \eta^{t+1}, x^{t+1} - x^{*} \rangle - \langle \nabla f(x^{t}), x^{t+1} - x^{t} \rangle + \langle \nabla f(x^{t}), x^{*} - x^{t} \rangle$$

For the first term in the above we can use the fact that $\eta^{t+1} \in \partial g(x^{t+1})$ to write

$$-\langle \eta^{t+1}, x^{t+1} - x^* \rangle = \langle \eta^{t+1}, x^* - x^{t+1} \rangle \le g(x^*) - g(x^{t+1}).$$
(102)

On the second term we can use the fact that f is L-smooth and (9) to write

$$-\langle \nabla f(x^{t}), x^{t+1} - x^{t} \rangle \le \frac{L}{2} \|x^{t+1} - x^{t}\|^{2} + f(x^{t}) - f(x^{t+1}).$$
(103)

On the last term we can use the convexity of f and (2) to write

$$\langle \nabla f(x^t), x^* - x^t \rangle \le f(x^*) - f(x^t).$$
(104)

By combining (102), (103), (104), and using the fact that $\gamma_t L \leq \gamma_0 L_{\text{max}} \leq 1$, we obtain

$$\frac{1}{2\gamma_t} \|x^{t+1} - x^*\|^2 - \frac{1}{2\gamma_t} \|x^t - x^*\|^2
\leq \frac{-1}{2\gamma_t} \|x^{t+1} - x^t\|^2 + \frac{L}{2} \|x^{t+1} - x^t\|^2 - (F(x^{t+1}) - \inf F) - \langle \nabla f_j(x^t) - \nabla f(x^t), x^{t+1} - x^* \rangle
\leq -(F(x^{t+1}) - \inf F) - \langle \nabla f_j(x^t) - \nabla f(x^t), x^{t+1} - x^* \rangle.$$
(105)

We now have to control the last term of (105), in expectation. To shorten the computation we temporarily introduce the operators

$$T \stackrel{\text{def}}{=} I_d - \gamma_t \nabla f,$$
$$\hat{T} \stackrel{\text{def}}{=} I_d - \gamma_t \nabla f_j.$$

Notice in particular that $x^{t+1} = \operatorname{prox}_{\gamma_t g}(\hat{T}(x^t))$. We have that

$$-\langle \nabla f_j(x^t) - \nabla f(x^t), x^{t+1} - x^* \rangle = -\langle \nabla f_j(x^t) - \nabla f(x^t), \operatorname{prox}_{\gamma_t g}(\hat{T}(x^t)) - \operatorname{prox}_{\gamma_t g}(T(x^t)) \rangle -\langle \nabla f_j(x^t) - \nabla f(x^t), \operatorname{prox}_{\gamma_t g}(T(x^t)) - x^* \rangle,$$
(106)

and observe that the last term is, in expectation, is equal to zero. This is due to the fact that $\operatorname{prox}_{\gamma_t g}(T(x^t)) - x^*$ is deterministic when conditioned on x^t . Since we will later on take expectations, we drop this term and keep on going. As for the first term, using the nonexpansiveness of the proximal operator (Lemma 8.16), we have that

$$\begin{aligned} -\langle \nabla f_j(x^t) - \nabla f(x^t), \operatorname{prox}_{\gamma_t g}(\hat{T}(x^t)) - \operatorname{prox}_{\gamma_t g}(T(x^t)) \rangle &\leq \|\nabla f_j(x^t) - \nabla f(x^t)\| \|\hat{T}(x^t) - T(x^t)\| \\ &= \gamma_t \|\nabla f_j(x^t) - \nabla f(x^t)\|^2 \end{aligned}$$

Using the above two bounds in (106) we have proved that (after taking expectation)

$$\mathbb{E}\left[-\langle \nabla f_j(x^t) - \nabla f(x^t), x^{t+1} - x^* \rangle\right] \le \gamma_t \mathbb{E}\left[\|\nabla f_j(x^t) - \nabla f(x^t)\|^2\right] = \gamma_t \mathbb{V}\left[\nabla f_j(x^t)\right].$$

Injecting the above inequality into (105), we finally obtain

$$\frac{1}{2}\mathbb{E}\left[\|x^{t+1} - x^*\|^2\right] - \frac{1}{2}\mathbb{E}\left[\|x^t - x^*\|^2\right] \le -\gamma_t \mathbb{E}\left[F(x^{t+1}) - \inf F\right] + \gamma_t^2 \mathbb{V}\left[\nabla f_i(x^t)\right].$$
(107)

To control the variance term $\mathbb{V}\left[\nabla f_i(x^t)\right]$ we use the variance transfer Lemma 8.20 with $x = x^t$ and $y = x^*$, which together with Definition 4.16 and Lemma 8.19 gives

$$\mathbb{V}\left[\nabla f_i(x^t) \mid x^t\right] \leq 4L_{\max}D_f(x^t;x^*) + 2\sigma_F^* \\ \leq 4L_{\max}\left(F(x^t) - \inf F\right) + 2\sigma_F^*.$$

Taking expectation in the above inequality and inserting it in (107) gives

$$\frac{1}{2}\mathbb{E}\left[\|x^{t+1} - x^*\|^2\right] - \frac{1}{2}\mathbb{E}\left[\|x^t - x^*\|^2\right]
\leq -\gamma_t \mathbb{E}\left[F(x^{t+1}) - \inf F\right] + 4\gamma_t^2 L_{\max} \mathbb{E}\left[F(x^t) - \inf F\right] + 2\gamma_t^2 \sigma_F^*.$$
(108)

Now we define the following Lyapunov energy for all $T\in\mathbb{N}$

$$E_T := \frac{1}{2} \mathbb{E} \left[\|x^T - x^*\|^2 \right] + \gamma_{T-1} \mathbb{E} \left[F(x^T) - \inf F \right] + \sum_{t=0}^{T-1} \gamma_t (1 - 4\gamma_t L_{\max}) \mathbb{E} \left[F(x^t) - \inf F \right] - 2\sigma_F^* \sum_{t=0}^{T-1} \gamma_t^2,$$

and we will show that it is decreasing (we note $\gamma_{-1} := \gamma_0$). Using the above definition we have that

$$E_{T+1} - E_T = \frac{1}{2} \mathbb{E} \left[\|x^{T+1} - x^*\|^2 \right] - \frac{1}{2} \mathbb{E} \left[\|x^T - x^*\|^2 \right] + \gamma_T \mathbb{E} \left[F(x^{T+1}) - \inf F \right] - \gamma_{T-1} \mathbb{E} \left[F(x^T) - \inf F \right] + \gamma_T (1 - 4\gamma_T L_{\max}) \mathbb{E} \left[F(x^T) - \inf F \right] - 2\sigma_F^* \gamma_T^2.$$

Using (108) and cancelling the matching terms, it remains

$$E_{T+1} - E_T \le -\gamma_{T-1} \mathbb{E}\left[F(x^T) - \inf F\right] + \gamma_T \mathbb{E}\left[F(x^T) - \inf F\right].$$

Using the fact that the stepsizes are nonincreasing $(\gamma_T \leq \gamma_{T-1})$ we consequently conclude that $E_{T+1} - E_T \leq 0$. Therefore, from the definition of E_T (and using again $\gamma_t \leq \gamma_0$), we have that

$$E_{0} = \frac{\|x^{0} - x^{*}\|^{2} + 2\gamma_{0}(F(x^{0}) - \inf F)}{2}$$

$$\geq E_{T} \geq \sum_{t=0}^{T-1} \gamma_{t}(1 - 4\gamma_{t}L_{\max})\mathbb{E}\left[F(x^{t}) - \inf F\right] - 2\sigma_{F}^{*}\sum_{t=0}^{T-1} \gamma_{t}^{2}$$

$$\geq (1 - 4\gamma_{0}L_{\max})\sum_{t=0}^{T-1} \gamma_{t}\mathbb{E}\left[F(x^{t}) - \inf F\right] - 2\sigma_{F}^{*}\sum_{t=0}^{T-1} \gamma_{t}^{2}.$$

Now passing the term in σ_f^* to the other side, dividing this inequality by $(1 - 4\gamma L_{\max}) \sum_{t=0}^{T-1} \gamma_t$, which is strictly positive since $4\gamma_0 L_{\max} < 1$, and using Jensen inequality, we finally conclude that

$$\mathbb{E}\left[F(\bar{x}^{T}) - \inf F\right] \leq \frac{1}{\sum_{t=0}^{T-1} \gamma_{t}} \sum_{t=0}^{T-1} \gamma_{t} \mathbb{E}\left[F(x^{t}) - \inf F\right] \\
\leq \frac{\|x^{0} - x^{*}\|^{2} + 2(F(x^{0}) - \inf F)}{2(1 - 4\gamma_{0}L_{\max})\sum_{t=0}^{T-1} \gamma_{t}} + \frac{2\sigma_{F}^{*} \sum_{t=0}^{T-1} \gamma_{t}^{2}}{(1 - 4\gamma_{0}L_{\max})\sum_{t=0}^{T-1} \gamma_{t}}.$$

Analogously to Remark 5.4, different choices for the step sizes γ_t allow us to trade off the convergence speed for the constant variance term. In the next two corollaries we choose a constant and a $1/\sqrt{t}$ step size, respectively, followed by a $\mathcal{O}(\epsilon^2)$ complexity result.

Corollary 11.6. Let Assumptions (Composite Sum of L_{max} -Smooth) and (Composite Sum of Convex) hold. Let $(x^t)_{t\in\mathbb{N}}$ be a sequence generated by the (SPGD) algorithm with a constant stepsize verifying $0 < \gamma < \frac{1}{4L_{\max}}$. Then, for all $x^* \in \operatorname{argmin} F$ and for all $t \in \mathbb{N}$ we have that

$$\mathbb{E}\left[F(\bar{x}^{t}) - \inf F\right] \leq \frac{\|x^{0} - x^{*}\|^{2} + 2\gamma(F(x^{0}) - \inf F)}{2(1 - 4\gamma L_{\max})\gamma t} + \frac{2\sigma_{F}^{*}\gamma}{(1 - 4\gamma L_{\max})},$$

where $\bar{x}^t \stackrel{\text{def}}{=} \frac{1}{T} \sum_{s=0}^{t-1} x^s$.

Proof. It is a direct consequence of Theorem 11.5 with $\gamma_t = \gamma$ and

$$\sum_{s=0}^{t-1} \gamma_s = \gamma t \quad \text{and} \quad \sum_{s=0}^{t-1} \gamma_s^2 = \gamma^2 t.$$

Corollary 11.7. Let Assumptions (Composite Sum of L_{max} -Smooth) and (Composite Sum of Convex) hold. Let $(x^t)_{t\in\mathbb{N}}$ be a sequence generated by the (SPGD) algorithm with a sequence of stepsizes $\gamma_t = \frac{\gamma_0}{\sqrt{t+1}}$ verifying $0 < \gamma_0 < \frac{1}{4L_{\text{max}}}$. Then, for all $t \ge 3$ we have that

$$\mathbb{E}\left[F(\bar{x}^t) - \inf F\right] = \mathcal{O}\left(\frac{\ln(t)}{\sqrt{t}}\right),\,$$

where $\bar{x}^t \stackrel{\text{def}}{=} \frac{1}{\sum_{s=0}^{t-1} \gamma_s} \sum_{s=0}^{t-1} \gamma_s x^s$.

Proof. Apply Theorem 11.5 with $\gamma_t = \frac{\gamma_0}{\sqrt{t+1}}$, and use the following integral bounds

$$\sum_{s=0}^{t-1} \gamma_s^2 = \gamma_0^2 \sum_{s=0}^{t-1} \frac{1}{s+1} \le \gamma_0^2 \int_{s=0}^{t-1} \frac{1}{s+1} ds = \gamma_0^2 \ln(t),$$

and

$$\sum_{s=0}^{t-1} \gamma_s \geq \int_{s=1}^{t-1} \frac{\gamma_0}{\sqrt{s+1}} = 2\gamma_0 \left(\sqrt{t} - \sqrt{2}\right).$$

For $t \ge 3$ we have $\left(\sqrt{t} - \sqrt{2}\right) > \frac{1}{6}\sqrt{t}$, and $\ln(t) > 1$, so we can conclude that

$$\frac{1}{\sum_{s=0}^{t-1} \gamma_s} \le \frac{1}{2\gamma_0(\sqrt{t} - \sqrt{2})} = \mathcal{O}\left(\frac{1}{\sqrt{t}}\right) = \mathcal{O}\left(\frac{\ln(t)}{\sqrt{t}}\right) \quad \text{and} \quad \frac{\sum_{s=0}^{t-1} \gamma_s^2}{\sum_{s=0}^{t-1} \gamma_s} = \mathcal{O}\left(\frac{\ln(t)}{\sqrt{t}}\right).$$

Even though constant stepsizes do not lead to convergence of the algorithm, we can guarantee arbitrary precision provided we take *small* stepsizes and a sufficient number of iterations.

Corollary 11.8 ($\mathcal{O}(\epsilon^{-2})$ -Complexity). Let Assumptions (Composite Sum of L_{\max} -Smooth) and (Composite Sum of Convex) hold. Let $(x^t)_{t\in\mathbb{N}}$ be a sequence generated by the (SPGD) algorithm with a constant stepsize γ . For every $0 < \varepsilon \leq \frac{\sigma_F^*}{L_{\max}}$, we can guarantee that $\mathbb{E}\left[F(\bar{x}^t) - \inf F\right] \leq \varepsilon$, provided that

$$\gamma = \frac{\varepsilon}{8\sigma_f^*} \quad \text{and} \quad t \ge C_0 \frac{\sigma_F}{\varepsilon^2},$$

e $\bar{x}^t = \frac{1}{t} \sum_{s=0}^{t-1} x^s, C_0 = 16 \left(\|x^0 - x^*\|^2 + \frac{1}{4L_{\max}} (F(x^0) - \inf F) \right) \text{ and } x^* \in \operatorname{argmin}$

Proof. First, note that our assumptions $\varepsilon \leq \frac{\sigma_F^*}{L_{\max}}$ and $\gamma = \frac{\varepsilon}{8\sigma_f^*}$ imply that $\gamma \leq \frac{1}{8L_{\max}}$, so the conditions of Corollary 11.6 hold true, and we obtain

$$\mathbb{E}\left[F(\bar{x}^{t}) - \inf F\right] \le A + B \quad \text{where} \quad A := \frac{\|x^{0} - x^{*}\|^{2} + 2\gamma(F(x^{0}) - \inf F)}{2(1 - 4\gamma L_{\max})\gamma t}, \ B := \frac{2\sigma_{F}^{*}\gamma}{(1 - 4\gamma L_{\max})^{2}}.$$

Passing the terms around, we see that

where

$$B \le \frac{\varepsilon}{2} \Leftrightarrow \gamma \le \frac{\varepsilon}{4(\sigma_F^* + \varepsilon L_{\max})}$$

Since we assumed that $\varepsilon \leq \frac{\sigma_F^*}{L_{\max}}$ and $\gamma = \frac{\varepsilon}{8\sigma_f^*}$, the above inequality is true. Now we note $h_0 := ||x^0 - x^*||^2$, $r_0 := F(x^0) - \inf F$, and write

$$A \le \frac{\varepsilon}{2} \Leftrightarrow t \ge \frac{h_0 + 2\gamma r_0}{2(1 - 4\gamma L_{\max})\gamma} \frac{2}{\varepsilon}.$$
(109)

With our assumptions, we see that

$$h_0 + 2\gamma r_0 \le h_0 + \frac{1}{4L_{\max}} r_0 = \frac{C_0}{16},$$
$$(1 - 4\gamma L_{\max})\gamma \varepsilon \ge \frac{1}{2} \frac{\varepsilon}{8\sigma_F^*} \varepsilon = \frac{\varepsilon^2}{16\sigma_F^*}$$

so indeed the inequality in (109) is true, which concludes the proof.

F.

11.2 Complexity for strongly convex functions

Theorem 11.9. Let Assumptions (Composite Sum of L_{\max} -Smooth) and (Composite Sum of Convex) hold, and assume further that f is μ -strongly convex, for $\mu > 0$. Let $(x^t)_{t \in \mathbb{N}}$ be a sequence generated by the (SPGD) algorithm with a constant sequence of stepsizes verifying $0 < \gamma \leq \frac{1}{2L_{\max}}$. Then, for $x^* = \operatorname{argmin} F$, for all $t \in \mathbb{N}$,

$$\mathbb{E}\left[\|x^{t} - x^{*}\|^{2}\right] \leq (1 - \gamma\mu)^{t} \|x^{0} - x^{*}\|^{2} + \frac{2\gamma\sigma_{F}^{*}}{\mu}.$$

Proof. In this proof, we fix $j := i_t$ to lighten the notations. Let us start by using the fixed-point property of the (PGD) algorithm (Lemma 8.17), together with the nonexpansiveness of the

proximal operator (Lemma 8.16), to write

$$\|x^{t+1} - x^*\|^2 = \|\operatorname{prox}_{\gamma g}(x^t - \gamma \nabla f_j(x^t)) - \operatorname{prox}_{\gamma g}(x^* - \gamma \nabla f(x^*))\|^2$$

$$\leq \|(x^t - \gamma \nabla f_j(x^t)) - (x^* - \gamma \nabla f(x^*))\|^2$$

$$= \|x^t - x^*\|^2 + \gamma^2 \|\nabla f_j(x^t) - \nabla f(x^*)\|^2 - 2\gamma \langle \nabla f_j(x^t) - \nabla f(x^*), x^t - x^* \rangle.$$
(110)

Let us analyse the last two terms of the right-hand side of (110). For the first term we use the Young's and the triangular inequality together with Lemma 8.20 we obtain

$$\gamma^{2} \mathbb{E}_{x^{t}} \left[\| \nabla f_{j}(x^{t}) - \nabla f(x^{*}) \|^{2} \right] \leq 2\gamma^{2} \mathbb{E}_{x^{t}} \left[\| \nabla f_{j}(x^{t}) - \nabla f_{j}(x^{*}) \|^{2} \right] + 2\gamma^{2} \mathbb{E} \left[\| \nabla f_{j}(x^{*}) - \nabla f(x^{*}) \|^{2} \right] \\
= 2\gamma^{2} \mathbb{E}_{x^{t}} \left[\| \nabla f_{j}(x^{t}) - \nabla f_{j}(x^{*}) \|^{2} \right] + 2\gamma^{2} \sigma_{F}^{*} \tag{111} \\
\leq 4\gamma^{2} L_{\max} D_{f}(x^{t}; x^{*}) + 2\gamma^{2} \sigma_{F}^{*},$$

where D_f is the divergence of f (see Definition 8.18). For the second term in (110) we use the strong convexity of f (Lemma 2.14) to write

$$-2\gamma \mathbb{E}_{x^{t}} \left[\langle \nabla f_{j}(x^{t}) - \nabla f(x^{*}), x^{t} - x^{*} \rangle \right] = -2\gamma \langle \nabla f(x^{t}) - \nabla f(x^{*}), x^{t} - x^{*} \rangle$$
$$= 2\gamma \langle \nabla f(x^{t}), x^{*} - x^{t} \rangle + 2\gamma \langle \nabla f(x^{*}), x^{t} - x^{*} \rangle$$
(112)
$$\leq -\gamma \mu \|x^{t} - x^{*}\|^{2} - 2\gamma D_{f}(x^{t}; x^{*}).$$

Combining (110), (111) and (112) and taking expectations gives

$$\mathbb{E} \left[\|x^{t+1} - x^*\|^2 \right] \leq (1 - \gamma \mu) \mathbb{E} \left[\|x^t - x^*\|^2 \right] + 2\gamma (2\gamma L_{\max} - 1) \mathbb{E} \left[D_f(x^t; x^*) \right] + 2\gamma^2 \sigma_F^* \\ \leq (1 - \gamma \mu) \mathbb{E} \left[\|x^t - x^*\|^2 \right] + 2\gamma^2 \sigma_F^*,$$

where in the last inequality we used our assumption that $2\gamma L_{\text{max}} \leq 1$. Now, recursively apply the above to write

$$\mathbb{E}\left[\|x^{t} - x^{*}\|^{2}\right] \leq (1 - \gamma\mu)^{t} \|x^{0} - x^{*}\|^{2} + 2\gamma^{2}\sigma_{F}^{*} \sum_{s=0}^{t-1} (1 - \gamma\mu)^{s},$$

and conclude by upper bounding this geometric sum using

$$\sum_{s=0}^{t-1} (1 - \gamma \mu)^s = \frac{1 - (1 - \gamma \mu)^t}{\gamma \mu} \le \frac{1}{\gamma \mu}.$$

Corollary 11.10 ($\tilde{\mathcal{O}}(1/\epsilon)$ Complexity). Consider the setting of Theorem 11.9. Let $\epsilon > 0$. If we set

$$\gamma = \min\left\{\frac{\epsilon}{2}\frac{\mu}{2\sigma_F^*}, \frac{1}{2L_{\max}}\right\}$$
(113)

then

$$t \ge \max\left\{\frac{1}{\epsilon}\frac{4\sigma_F^*}{\mu^2}, \ \frac{2L_{\max}}{\mu}\right\}\log\left(\frac{2\|x^0 - x^*\|^2}{\epsilon}\right) \implies \|x^t - x^*\|^2 \le \epsilon \tag{114}$$

Proof. Applying Lemma A.2 with $A = \frac{2\sigma_F^*}{\mu}$, $C = 2L_{\text{max}}$ and $\alpha_0 = ||x^0 - x^*||^2$ gives the result (49).

11.3 Bibliographic notes

In proving Theorem 11.5 we simplified the more general Theorem 3.3 in [21], which gives a convergence rate for several stochastic proximal algorithms for which proximal SGD is one special case. In the smooth and strongly convex case the paper [13] gives a general theorem for the convergence of stochastic proximal algorithms which includes proximal SGD. For adaptive stochastic proximal methods see [1].

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A Appendix

A.1 Lemmas for Complexity

The following lemma was copied from Lemma 11 in [14].

Lemma A.1. Consider the sequence $(\alpha_k)_k \in \mathbb{R}_+$ of positive scalars that converges to zero according to

$$\alpha_k \le \rho^k \, \alpha_0, \tag{115}$$

where $\rho \in [0, 1)$. For a given $1 > \epsilon > 0$ we have that

$$k \ge \frac{1}{1-\rho} \log\left(\frac{1}{\epsilon}\right) \quad \Rightarrow \quad \alpha_k \le \epsilon \,\alpha_0.$$
 (116)

Proof. First note that if $\rho = 0$ the result follows trivially. Assuming $\rho \in (0, 1)$, rearranging (115) and applying the logarithm to both sides gives

$$\log\left(\frac{\alpha_0}{\alpha_k}\right) \ge k \log\left(\frac{1}{\rho}\right). \tag{117}$$

Now using that

$$\frac{1}{1-\rho}\log\left(\frac{1}{\rho}\right) \ge 1,\tag{118}$$

for all $\rho \in (0, 1)$ and assuming that

$$k \ge \frac{1}{1-\rho} \log\left(\frac{1}{\epsilon}\right),\tag{119}$$

we have that

$$\log \left(\frac{\alpha_0}{\alpha_k}\right) \stackrel{(117)}{\geq} k \log \left(\frac{1}{\rho}\right)$$
$$\stackrel{(119)}{\geq} \frac{1}{1-\rho} \log \left(\frac{1}{\rho}\right) \log \left(\frac{1}{\epsilon}\right)$$
$$\stackrel{(118)}{\geq} \log \left(\frac{1}{\epsilon}\right)$$

Applying exponentials to the above inequality gives (116).

As an example of the use this lemma, consider the sequence of random vectors $(Y^k)_k$ for which the expected norm converges to zero according to

$$\mathbb{E}\left[\|Y^{k}\|^{2}\right] \le \rho^{k} \|Y^{0}\|^{2}.$$
(120)

Then applying Lemma A.1 with $\alpha_k = \mathbb{E}\left[\|Y^k\|^2 \right]$ for a given $1 > \epsilon > 0$ states that

$$k \ge \frac{1}{1-\rho} \log\left(\frac{1}{\epsilon}\right) \quad \Rightarrow \quad \mathbb{E}\left[\|Y^k\|^2\right] \le \epsilon \, \|Y^0\|^2.$$

Lemma A.2. Consider the recurrence given by

$$\alpha_k \le (1 - \gamma \mu)^t \alpha_0 + A\gamma, \tag{121}$$

where $\mu > 0$ and $A, C \ge 0$ are given constants and $\gamma \le \frac{1}{C}$. If

$$\gamma = \min\left\{\frac{\epsilon}{2A}, \ \frac{1}{C}\right\} \tag{122}$$

then

$$t \ge \max\left\{\frac{1}{\epsilon}\frac{2A}{\mu}, \frac{C}{\mu}\right\}\log\left(\frac{2\alpha_0}{\epsilon}\right) \implies \alpha_k \le \epsilon.$$

Proof. First we restrict γ so that the second term in (121) is less than $\epsilon/2$, that is

$$A\gamma \leq \frac{\epsilon}{2} \implies \gamma \leq \frac{\epsilon}{2A}.$$

Thus we set γ according to (122) to also satisfy the constraint that $\gamma \leq \frac{1}{C}$.

Furthermore we want

$$(1-\mu\gamma)^t\alpha_0 < \frac{\epsilon}{2},$$

then taking logarithms and re-arranging the above gives

$$\log\left(\frac{2\alpha_0}{\epsilon}\right) \le t \log\left(\frac{1}{1-\gamma\mu}\right). \tag{123}$$

Now using that $\log\left(\frac{1}{\rho}\right) \ge 1 - \rho$, for $0 < \rho \le 1$ with $\rho = 1 - \gamma \mu$ gives

$$t \ge \frac{1}{\mu\gamma} \log\left(\frac{2\alpha_0}{\epsilon}\right).$$

Substituting in γ from (122) gives

$$t \ge \max\left\{\frac{1}{\epsilon}\frac{2A}{\mu}, \frac{C}{\mu}\right\}\log\left(\frac{2\alpha_0}{\epsilon}\right)$$

A.2 A nonconvex PŁ function

Lemma A.3 (A nonconvex PL function). Let $f(t) = t^2 + 3\sin(t)^2$. Then f is μ -Polyak-Lojasiewicz with $\mu = \frac{1}{40}$, while not being convex.

Proof. The fact that f is not convex follows directly from the fact that $f''(t) = 2 + 6\cos(2t)$ can be nonpositive, for instance $f''(\frac{\pi}{2}) = -4$. To prove that f is PL, start by computing $f'(t) = 2t + 3\sin(2t)$, and $\inf f = 0$. Therefore we are looking for a constant $\alpha = 2\mu > 0$ such that

for all
$$t \in \mathbb{R}$$
, $(2t+3\sin(2t))^2 \ge \alpha \left(t^2+3\sin(t)^2\right)$.

Using the fact that $\sin(t)^2 \le t^2$, we see that it is sufficient to find $\alpha > 0$ such that

for all $t \in \mathbb{R}$, $(2t + 3\sin(2t))^2 \ge 4\alpha t^2$.

Now let us introduce X = 2t, $Y = 3\sin(2t)$, so that the above property is equivalent to

for all
$$(X, Y) \in \mathbb{R}^2$$
 such that $Y = 3\sin(X)$, $(X + Y)^2 \ge \alpha X^2$.

It is easy to check whenever the inequality $(X + Y)^2 \ge \alpha X^2$ is verified or not:

$$(X+Y)^2 < \alpha X^2 \Leftrightarrow \begin{cases} X > 0 \text{ and } -(1+\sqrt{\alpha})X < Y < -(1-\sqrt{\alpha})X \\ \text{or} \\ X < 0 \text{ and } -(1-\sqrt{\alpha})X < Y < -(1+\sqrt{\alpha})X \end{cases}$$
(124)

Now we just need to make sure that the curve $Y = 3\sin(X)$ violates those conditions for α small enough. We will consider different cases depending on the value of X:

- If $X \in [0, \pi]$, we have $Y = 3\sin(X) \ge 0 > -(1 \sqrt{\alpha})X$, provided that $\alpha < 1$.
- On $[\pi, \frac{5}{4}\pi]$ we can use the inequality $\sin(t) \ge \pi t$. One way to prove this inequality is to use the fact that $\sin(t)$ is convex on $[\pi, 2\pi]$ (its second derivative is $-\sin(t) \ge 0$), which implies that $\sin(t)$ is greater than its tangent at $t_0 = \pi$, whose equation is $\pi - t$. This being said, we can write (remember that $X \in [\pi, \frac{5}{4}\pi]$ here):

$$Y = 3\sin(X) \ge 3(\pi - X) \ge 3(\pi - \frac{5}{4}\pi) = -\frac{3}{4}\pi > -(1 - \sqrt{\alpha})\pi \ge -(1 - \sqrt{\alpha})X,$$

where the strict inequality is true whenever $\frac{3}{4} < 1 - \sqrt{\alpha} \Leftrightarrow \alpha < \frac{1}{16} \simeq 0.06$.

• If $X \in [\frac{5}{4}\pi, +\infty[$, we simply use the fact that

$$Y = 3\sin(X) \ge -3 > -(1 - \sqrt{\alpha})\frac{5}{4}\pi \ge -(1 - \sqrt{\alpha})X,$$

where the strict inequality is true whenever $3 < (1 - \sqrt{\alpha})\frac{5}{4}\pi \Leftrightarrow \alpha < (1 - \frac{12}{5\pi})^2 \simeq 0.055$.

If X ∈]−∞, 0], we can use the exact same arguments (use the fact that sine is a odd function) to obtain that Y < −(1 − √α)X.

In every cases, we see that (124) is violated when $Y = 3\sin(X)$ and $\alpha = 0.05$, which allows us to conclude that f is μ -PL with $\mu = \alpha/2 = 0.025 = 1/40$.